UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
GR II (Phys. 750) — Prof. Leo C. Stein — Spring 2020

Problem Set 6 — SOLUTIONS

Due: Thurs., Apr. 16, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Abstract 3+1 basics**. Let $(\mathcal{M}, g_{ab})$ be a 4-manifold with Lorentzian metric g (with Levi-Civita connection ∇), and suppose we have a foliation by hypersurfaces of a global "time" function $t : \mathcal{M} \to \mathbb{R}$, each leaf being a spacelike hypersurface Σ_t. Let the future-pointing, timelike, unit normal be

$$n^a \equiv -\alpha \nabla^a t = \frac{-\nabla^a t}{\sqrt{-g^{ab}\nabla_a t \nabla_b t}}\,, \tag{1}$$

where the lapse α is defined through $g^{ab}\nabla_a t \nabla_b t = -1/\alpha^2$.

(a) Show that the one-form field n_a is *irrotational*, which is to say that it satisfies $n_{[a}\nabla_b n_{c]} = 0$.
Solution: Expand all the definitions:

$$\begin{aligned}6n_{[a}\nabla_b n_{c]} = \alpha[&(\nabla_a t)\nabla_b(\alpha\nabla_c t) + (\nabla_b t)\nabla_c(\alpha\nabla_a t) + (\nabla_c t)\nabla_a(\alpha\nabla_b t)\\ &- (\nabla_a t)\nabla_c(\alpha\nabla_b t) - (\nabla_b t)\nabla_a(\alpha\nabla_c t) - (\nabla_c t)\nabla_b(\alpha\nabla_a t)]\,.\end{aligned} \tag{2}$$

Now apply the product rule, and using the fact that the ∇ connection is torsion free, we have that $\nabla_a\nabla_b t = \nabla_b\nabla_a t$. Match up antisymmetric combinations of the second derivatives of t and they cancel. So we are left with

$$\begin{aligned}6n_{[a}\nabla_b n_{c]} &= \alpha[(\nabla_a t)(\nabla_b\alpha)\nabla_c t + (\nabla_b t)(\nabla_c\alpha)\nabla_a t + (\nabla_c t)(\nabla_a\alpha)\nabla_b t\\ &\qquad - (\nabla_a t)(\nabla_c\alpha)\nabla_b t - (\nabla_b t)(\nabla_a\alpha)\nabla_c t - (\nabla_c t)(\nabla_b\alpha)\nabla_a t] \qquad (3)\\ &= 6\alpha(\nabla_{[a}t)(\nabla_b\alpha)(\nabla_{c]}t) = 0\,, \qquad (4)\end{aligned}$$

since the tensor product $(\nabla_a t)(\nabla_c t)$ is symmetric, but is being antisymmetrized.

Now recall that we decompose each tangent space into the subspace $T\Sigma_t$ and the orthogonal complement in the span of n^a, by writing the induced metric γ_{ab} on $T\Sigma_t$ (sometimes called the "first fundamental form"),

$$\gamma_{ab} = g_{ab} + n_a n_b\,. \tag{5}$$

Then the (1,1) version $\gamma^a{}_b$ is an (idempotent) projection operator. The induced metric γ_{ab} has Levi-Civita connection D_a. This connection only knows how to act on purely spatial tensors – recall that a tensor is purely spatial iff it vanishes when n^a is contracted into any index slot. The simple way to define D_a is

$$D_c T^{a_1 a_2\cdots}{}_{b_1\cdots} = \gamma_c^{c'}\gamma^{a_1}_{a'_1}\cdots\gamma^{b'_1}_{b_1}\cdots\nabla_{c'}T^{a'_1 a'_2\cdots}{}_{b'_1\cdots}\,. \tag{6}$$

(b) Show that D_a is indeed metric-compatible with γ_{bc}.
Solution: We want to show that $D_a\gamma_{bc} = 0$. Expanding the definition, use metric-compatibility of g and ∇, then use the Leibniz rule,

$$\begin{aligned}D_a\gamma_{bc} &= \gamma_a^{a'}\gamma_b^{b'}\gamma_c^{c'}\nabla_{a'}(g_{b'c'} + n_{b'}n_{c'}) = \gamma_a^{a'}\gamma_b^{b'}\gamma_c^{c'}\nabla_{a'}(n_{b'}n_{c'}) \qquad (7)\\ &= \gamma_a^{a'}\gamma_b^{b'}\gamma_c^{c'}[(\nabla_{a'}n_{b'})n_{c'} + n_{b'}(\nabla_{a'}n_{c'})] = 0\,. \qquad (8)\end{aligned}$$

The resulting terms on the RHS contain either $\gamma_b^{b'}n_{b'} = 0$ or $\gamma_c^{c'}n_{c'} = 0$, so both terms vanish.

(c) Show that the Leibniz rule $D_a(v^b w_b) = v^b(D_a w_b) + (D_a v^b)w_b$ holds only if v^b and w_b are purely spatial.

Solution: Suppose v^a is not purely spatial, then we write it as $v^a = v_\parallel n^a + v_\perp^a$, where $v_\perp^a = \gamma^a_{a'} v_\perp^{a'}$ is perpendicular to n^a, and $v_\parallel \equiv -v_a n^a$ is a scalar. Then using the definition of D_a, we would find

$$D_a(v^b w_b) = \gamma_a^{a'} \nabla_{a'}[v_\parallel n^b w_b + v_\perp^b w_b] = \gamma_a^{a'} \nabla_{a'}[v_\perp^b w_b] = \gamma_a^{a'}[v_\perp^b(\nabla_{a'} w_b) + (\nabla_{a'} v_\perp^b) w_b] \tag{9}$$

$$= v_\perp^b(D_a w_b) + (D_a v^b) w_b \neq v^b(D_a w_b) + (D_a v^b) w_b \,. \tag{10}$$

In going from (9) to (10), we used the fact that $v_\perp^a = \gamma^a_{a'} v_\perp^{a'}$ and similarly $w^a = \gamma^a_{a'} w^{a'}$, then noting that the γ projectors are acting on all free indices of a ∇ derivative, thus giving a D derivative. If w^a was also not purely spatial, there would be an additional $v_\parallel w_\parallel$ term to consider, with the algebra working the same way. So we see that the Leibniz rule only works if both vectors are purely spatial.

If an observer was moving along a world-line with tangent n^a, then her proper acceleration 4-vector would be $a^a \equiv n^b \nabla_b n^a$.

(d) Show that the acceleration vector is a purely spatial vector.

Solution: Since the magnitude of n^b is constant, $n_b n^b = -1$,

$$0 = \nabla_a(n_b n^b) = 2 n_b \nabla_a n^b \,. \tag{11}$$

Thus the acceleration vector $a^b \equiv n^a \nabla_a n^b$ vanishes if its index is contracted with the timelike unit normal, $n_b a^b = n_b n^a \nabla_a n^b = 0$. This is the criterion for being purely spatial.

(e) Show that the acceleration can be written in terms of the spatial gradient of the lapse function,

$$a_a = D_a \ln \alpha \,. \tag{12}$$

Solution: As an intermediate result, let's compute the 4-dimensional gradient of $\alpha = [-(\nabla^c t)(\nabla_c t)]^{-1/2}$,

$$\nabla_a \alpha = \frac{-1}{2} \alpha^3 \nabla_a[-(\nabla^c t)(\nabla_c t)] = \alpha^3 (\nabla^c t) \nabla_a \nabla_c t \,. \tag{13}$$

Next for the spatial derivative of $\ln \alpha$,

$$D_a \ln \alpha = \gamma_a^{a'} \frac{1}{\alpha} \nabla_{a'} \alpha \tag{14}$$

$$D_a \ln \alpha = \left[\delta_a^{a'} + \alpha^2 (\nabla_a t)(\nabla^{a'} t)\right] \alpha^2 (\nabla^c t)(\nabla_{a'} \nabla_c t) \,. \tag{15}$$

Now let's compare with the expansion of $a_a \equiv n^{a'} \nabla_{a'} n_a$,

$$a_a \equiv n^{a'} \nabla_{a'} n_a = \alpha(\nabla^{a'} t) \nabla_{a'}(\alpha \nabla_a t) = \alpha(\nabla^{a'} t)\left[(\nabla_{a'} \alpha)(\nabla_a t) + \alpha \nabla_{a'} \nabla_a t\right] \tag{16}$$

$$a_a = \alpha(\nabla^{a'} t)\left[(\nabla_{a'} \alpha)(\nabla_a t) + \alpha \nabla_{a'} \nabla_a t\right] = \alpha(\nabla^{a'} t)\left[\alpha^3 (\nabla^c t)(\nabla_{a'} \nabla_c t)(\nabla_a t) + \alpha \nabla_{a'} \nabla_a t\right] \,. \tag{17}$$

Comparing the right-hand sides of (15) and (17), and using the fact that ∇ is torsion-free, we see that the two expressions are the same.

Now we're interested in the extrinsic curvature (sometimes called the "second fundamental form"), found by studying how n^a varies from point to point. Our convention is

$$K_{ab} \equiv -\gamma_a^c \nabla_c n_b \,, \tag{18}$$

which is a purely spatial tensor.

(f) To ensure this is purely spatial we did not need a γ projector on the b index – show why.

Solution: An index b of a tensor is a spatial index if the tensor vanishes when contracted with n^b, so let's check:

$$n^b K_{ab} = -\gamma_a^c n^b \nabla_c n_b = -\gamma_a^c \frac{1}{2} \nabla_c (n^b n_b) = 0\,, \tag{19}$$

since $n^b n_b = -1$ is a constant.

(g) Show the equality $K_{ab} = -\nabla_a n_b - n_a a_b$.

Solution: Expanding the definition,

$$K_{ab} \equiv -\gamma_a^c \nabla_c n_b = -(\delta_a^c + n_a n^c)\nabla_c n_b = -\nabla_a n_b - n_a (n^c \nabla_c n_b) = -\nabla_a n_b - n_a a_b\,, \tag{20}$$

where the last equality comes from the definition of $a_b \equiv n^c \nabla_c n_b$.

(h) Show that K_{ab} is a symmetric tensor.

Solution: Let's return to part 1a, where we showed that $n_{[a} \nabla_b n_{c]} = 0$. Contract this identity with n^a:

$$0 = 6 n_{[a} \nabla_b n_{c]} = n^a \left(n_a \nabla_b n_c + n_b \nabla_c n_a + n_c \nabla_a n_b - n_a \nabla_c n_b - n_b \nabla_a n_c - n_c \nabla_b n_a c \right) \tag{21}$$

$$0 = -\nabla_b n_c + 0 + n_c a_b + \nabla_c n_b - n_b a_c - 0\,, \tag{22}$$

where we have used the fact that $n^a \nabla_b n_a = 0$ because $n^a n_a = -1$ is a constant, and replacing $n^a \nabla_a n_b = a_b$ from the definition of the acceleration vector. But Eq. (22) is simply the expansion of $2K_{[ab]}$. So, we have shown that the antisymmetric part of K_{ab} vanishes. Since $K_{ab} = K_{(ab)} + K_{[ab]}$, we have shown that $K_{ab} = K_{(ab)}$, the extrinsic curvature is symmetric.

(i) Show the equality $K_{ab} = -\frac{1}{2} \mathcal{L}_{\boldsymbol{n}} \gamma_{ab}$.

Solution: Expanding the Lie derivative in terms of the ∇ connection,

$$\begin{aligned}
-\frac{1}{2} \mathcal{L}_{\boldsymbol{n}} \gamma_{ab} &= -\frac{1}{2} \left[n^c \nabla_c \gamma_{ab} + 2 \gamma_{c(a} \nabla_{b)} n^c \right] && (23)\\
&= -\frac{1}{2} n^c \nabla_c (g_{ab} + n_a n_b) - \gamma_{c(a} \nabla_{b)} n^c && (24)\\
&= -\frac{1}{2} n^c (\nabla_c n_a) n_b - \frac{1}{2} n^c n_a (\nabla_c n_b) - g_{c(a} \nabla_{b)} n^c + n_c n_{(a} \nabla_{b)} n^c && (25)\\
&= -\frac{1}{2} a_a n_b - \frac{1}{2} n_a a_b - g_{c(a} \nabla_{b)} n^c + 0 && (26)\\
&= -\frac{1}{2} a_a n_b - \frac{1}{2} n_a a_b - \frac{1}{2} \nabla_b n_a - \frac{1}{2} \nabla_a n_b = K_{(ab)} = K_{ab}\,. && (27)
\end{aligned}$$

(j) Show that $\mathcal{L}_{\boldsymbol{n}} K_{ab}$ is purely spatial.

Solution: The slick way to do this is as follows. Since K_{ab} is purely spatial, $n^a K_{ab} = 0$. Now apply $\mathcal{L}_{\boldsymbol{n}}$ to this latter equality, and expand using the Leibniz rule,

$$0 = \mathcal{L}_{\boldsymbol{n}} (n^a K_{ab}) = n^a (\mathcal{L}_{\boldsymbol{n}} K_{ab}) + (\mathcal{L}_{\boldsymbol{n}} n^a) K_{ab}\,. \tag{28}$$

In the second term, we use the fact that $\mathcal{L}_{\boldsymbol{v}} w^a = [v, w]^a$ for the Lie derivative of a vector. So here we get $[n, n]$, which vanishes by antisymmetry of the Lie bracket. Thus we have found $0 = n^a \mathcal{L}_{\boldsymbol{n}} K_{ab}$, which means that $\mathcal{L}_{\boldsymbol{n}} K_{ab}$ is purely spatial.

2. **A coordinate example**. Take the Schwarzschild spacetime in standard Schwarzschild coordinates. Define a foliation by level sets of the function

$$T = t + 4M\left[\sqrt{\frac{r}{2M}} + \frac{1}{2}\ln\left(\frac{\sqrt{r/2M}-1}{\sqrt{r/2M}+1}\right)\right] . \tag{29}$$

   (a) Compute the unit normal n_a and the induced metric γ_{ab}.

   (b) Calculate the extrinsic curvature K_{ab}.

   (c) Show that the Schwarzschild metric can be rewritten using T as a time coordinate instead of t, resulting in:

$$ds^2 = -dT^2 + (dr + \sqrt{2M/r}\; dT)^2 + r^2 d\Omega^2 . \tag{30}$$

   From this calculation you can see that the $T =$ const. surfaces are intrinsically flat.

   Solution: See the Mathematica notebook.

3. **A 4+1 example**. Let's start from 5-dimensional Minkowski space with coordinates z^A,

$$ds^2 = \eta_{AB} dz^A dz^B = -(dz^0)^2 + (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 . \tag{31}$$

   We construct a map from a 4-dimensional manifold with coordinates $x^a = (t, \chi, \theta, \phi)$ into this 5-d manifold, thus defining a 4-d hypersurface in 5-d. The coordinate maps for embedding are $z^A(x^a)$:

$$z^0 = a\sinh(t/a) , \quad z^1 = a\cosh(t/a)\cos\chi , \quad z^2 = a\cosh(t/a)\sin\chi\cos\theta , \tag{32}$$

$$z^3 = a\cosh(t/a)\sin\chi\sin\theta\cos\phi , \quad z^4 = a\cosh(t/a)\sin\chi\sin\theta\sin\phi . \tag{33}$$

   (a) Find a single coordinate function $\Phi(z^A)$ such that $\Phi = 0$ defines the same submanifold.
   Solution: Define the function

$$\sigma(z) \equiv \frac{1}{2}\eta_{AB} z^A z^B , \tag{34}$$

   which is half the squared geodesic distance from the origin to z (this is Synge's world function with argument at the origin). Let $\mathcal{S}$ be the image of the embedding above. Then notice that when evaluating σ on $\mathcal{S}$, we find

$$\sigma\Big|_{\mathcal{S}} = \frac{a^2}{2} . \tag{35}$$

   So, a function Φ that vanishes on $\mathcal{S}$ can be built out of σ. The simplest such function is

$$\Phi = \sigma - \frac{a^2}{2} . \tag{36}$$

   (b) Compute the unit normal n^A and the tangent vectors $e^A_{(a)} = \partial z^A/\partial x^a$.
   Solution: Computing the unit normal is straightforward,

$$(d\Phi)_A = \eta_{AB} z^B , \tag{37}$$

$$n^A = \frac{\varepsilon \eta^{AB}(d\Phi)_B}{\sqrt{\varepsilon \eta^{BC}(d\Phi)_B (d\Phi)_C}} = \frac{\varepsilon z^A}{\sqrt{\varepsilon 2\sigma}} , \tag{38}$$

   where $\varepsilon = \pm 1$ is the norm n, $\varepsilon = +1$ when n is spacelike, and $\varepsilon = -1$ when n is timelike. Here $a^2 > 0$ which makes n spacelike, so $\varepsilon = +1$. Evaluated on the submanifold we get

$$n^A\Big|_{\mathcal{S}} = \frac{1}{a} z^A . \tag{39}$$

Taking partial derivatives of the embedding functions, we get the coordinate tangent vectors spanning the submanifold surface as the four columns of this matrix, in the order $\partial_t, \partial_\chi, \partial_\theta, \partial_\phi$,

$$\begin{pmatrix}
\cosh(\frac{t}{a}) & 0 & 0 & 0 \\
\sinh(\frac{t}{a})\cos\chi & -a\cosh(\frac{t}{a})\sin\chi & 0 & 0 \\
\sinh(\frac{t}{a})\sin\chi\cos\theta & a\cosh(\frac{t}{a})\cos\chi\cos\theta & -a\cosh(\frac{t}{a})\sin\chi\sin\theta & 0 \\
\sinh(\frac{t}{a})\sin\chi\sin\theta\cos\phi & a\cosh(\frac{t}{a})\cos\chi\sin\theta\cos\phi & a\cosh(\frac{t}{a})\sin\chi\cos\theta\cos\phi & -a\cosh(\frac{t}{a})\sin\chi\sin\theta\sin\phi \\
\sinh(\frac{t}{a})\sin\chi\sin\theta\sin\phi & a\cosh(\frac{t}{a})\cos\chi\sin\theta\sin\phi & a\cosh(\frac{t}{a})\sin\chi\cos\theta\sin\phi & a\cosh(\frac{t}{a})\sin\chi\sin\theta\cos\phi
\end{pmatrix}$$

(c) Compute the induced 4-metric $\gamma_{ab} = \eta_{AB} e^A_a e^B_b$, and comment on the physical meaning of this metric.

Solution: Performing the contraction we find, in traditional notation,

$$\gamma_{ab} dx^a dx^b = -dt^2 + a^2 \cosh^2(t/a) \left[d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] . \tag{40}$$

This is a special case of the FLRW metric where the surfaces of $t =$ const. are 3-spheres of radius $a\cosh(t/a)$ (which is also the FLRW scale factor). It is one coordinate chart on the de Sitter metric.

(d) Compute the extrinsic curvature K_{ab}, then use the Gauss-Codazzi equations to show that the 4-metric is a metric of constant curvature,

$$^{(4)}R_{abcd} = \frac{1}{a^2}(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}) . \tag{41}$$

Solution: In terms of the unit normal and the basis e^A_a, we can write the pullback of the extrinsic curvature as

$$K_{ab} = -e^A_a e^B_b \nabla_A n_B . \tag{42}$$

(Notice that this sign convention agrees with MTW and Baumgarte+Shapiro, but differs from Poisson). Since our ambient space is Minkowski with flat rectangular coordinates, the covariant derivative is simply a partial derivative. Taking the partial derivatives, we have (using the notation $z_A \equiv \eta_{AB} z^B$)

$$\partial_A n_B = \partial_A \frac{z_B}{\sqrt{2\sigma}} = \frac{1}{\sqrt{2\sigma}} \partial_A z_B + z_B \partial_A \frac{1}{\sqrt{2\sigma}} \tag{43}$$

$$= \frac{1}{\sqrt{2\sigma}} \eta_{AB} - \frac{1}{2}\frac{1}{\sqrt{2}} z_B \sigma^{-3/2} \partial_A \sigma \tag{44}$$

$$\partial_A n_B = \frac{1}{\sqrt{2\sigma}} \left(\eta_{AB} - n_A n_B \right) . \tag{45}$$

Contracting with the bases, we find that the extrinsic curvature happens to be proportional to the 4-metric,

$$K_{bc} = \frac{-1}{a} \gamma_{bc} . \tag{46}$$

Now we will use the Gauss equation (notice that the MTW and Poisson sign conventions give the same result here),

$$^{(5)}R_{ABCD} e^A_a e^B_b e^C_c e^D_d = {}^{(4)}R_{abcd} + \varepsilon (K_{ad} K_{bc} - K_{ac} K_{bd}) . \tag{47}$$

Here, $^{(5)}R_{ABCD} = 0$ since the ambient space is 5-d Minkowski; and we have $\varepsilon = +1$. Therefore we find the desired result

$$^{(4)}R_{abcd} = -\frac{1}{a^2} (\gamma_{ad}\gamma_{bc} - \gamma_{ac}\gamma_{bd}) . \tag{48}$$

This is the curvature tensor of a maximally symmetric space.