UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
GR II (Phys. 750) — Prof. Leo C. Stein — Spring 2020

Problem Set 5

Due: Thurs., Apr. 02, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. In class, we derived the formula for the leading gravitational radiation generated by a source,

$$h_{ij}^{\mathrm{TT}}(t,r) = \frac{2}{r}\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right)\ddot{I}_{kl}(t-r)\,, \tag{1}$$

   where an overdot denotes derivative with respect to $u = t-r$, the second mass moment is $I_{kl} = \int \rho x^k x^l dV$, and the orthogonal projection tensor was $P_{ij} = \delta_{ij} - n_i n_j$ with $n^i = x^i/r$.

   Show that we get the same result if we replace I_{ij} with the tracefree mass quadrupole tensor, $\mathcal{I}\!\!\!-_{kl} \equiv I_{kl} - \frac{1}{3}\delta_{kl}I$, where $I = \delta_{ij}I_{ij}$ is the trace.

   Note that this is not trivial: there are two different types of trace removal. One is a three-dimensional trace, the other a 2-dimensional trace in the space orthogonal to n^i.

   The significance here is that while I has 6 components, $\mathcal{I}\!\!\!-$ only has 5 independent components, which is the correct number for a radiative quadrupole (and l-pole should have $2l+1$ components).

2. **Circular binary** Let's consider a circular binary with two point particle components of masses m_1, m_2 in a circular orbit lying in the $x-y$ plane. Let the separation be R, and to start we'll take the orbit to be Newtonian. Now let's compute the gravitational waves and the backreaction on the orbit. [Hint: everything will be simpler in terms of a reduced mass $\mu = m_1 m_2/(m_1+m_2)$ going around a central body of the total mass $M = m_1 + m_2$.]

   (a) Compute the gravitational wave tensor h_{ij}^{TT} at a point on the z axis.

   (b) Compute the energy loss due to gravitational waves (integrating over all emission directions). Remember that this can only be interpreted correctly when averaged over a few cycles of the radiation.

   (c) Now claiming that energy is conserved,

$$\frac{d}{dt}(E_{\mathrm{kin.}} + E_{\mathrm{pot.}} + E_{\mathrm{GW}}) = 0\,, \tag{2}$$

   derive an equation for how the orbit must shrink, dR/dt. [Hint: $E_{\mathrm{kin.}} + E_{\mathrm{pot.}}$ combine into a very simple expression of R for a bound orbit.]

   (d) Derive the rate of change of orbital frequency Ω caused by emission of GWs. You should get something in terms of the chirp mass, $\mathcal{M} \equiv \mu^{3/5}M^{2/5}$, to some power.

   (e) Integrate $d\Omega/dt$ to find the solution for $\Omega(t)$. This will diverge at some coalescence time $T_{\mathrm{coal.}}$ (this is an artifact of the point particle treatment of the bodies). Your solution should be some power law for $(T_{\mathrm{coal.}} - t)$.

3. **Wave equation for Riemann**. While a metric perturbation is not gauge invariant, the linearized Riemann tensor is when we're on a flat background (do you remember why?). So, let's get a wave equation for the Riemann tensor itself.

(a) Starting from the Einstein equations and the full Bianchi identity,

$$\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0\,, \tag{3}$$

derive an equation for some appropriate divergence,

$$\nabla_a R^a{}_{bcd} = 8\pi G[\text{sources involving one derivative of } T]\,. \tag{4}$$

(b) Now again starting from the Bianchi identity, derive an equation for

$$\Box R_{abcd} = 8\pi G[\text{sources built from two derivatives of } T\text{, and terms quadratic in } R]\,. \tag{5}$$

Here you will also make use of the result you got in item 3a. Note that there is a lot of room for error in the algebra for this problem. You may want to only work at the linearized level around a flat background, but I encourage you to work the full problem.

(c) Now specialize to a plane wave on a flat background, so that the curvature wave has the form $R_{abcd} = R_{abcd}(t - z)$. Use the Bianchi identities and symmetries of Riemann to show that the only independent components are R_{0i0j} (and others related by symmetries). [Hint: because we're on a flat background and linearizing, you can use a Fourier expansion and work mode by mode, using as an ansatz $R_{abcd} = C_{abcd}\exp(ik_e x^e)$ with a null wave-vector k_e pointing in the z direction, and some constant polarization tensor C].

(d) Show that in the above curvature wave propagating in the z direction, the only nonvanishing components are $R_{0x0x} = -R_{0y0y}$, and R_{0x0y}.

4. Contracting the Bianchi identities leads to the fact that the Einstein tensor is divergence-free, $\nabla_a G^a{}_b = 0$. Use this to *show* that $G^0{}_\mu$ must have fewer time derivatives than $G^i{}_\mu$. Thus conclude that the components $G^0{}_0$ and $G^0{}_i$ will be "constraint equations," while $G^i{}_j$ will be evolution equations that contain two time derivatives.