**UNIVERSITY OF MISSISSIPPI**
Department of Physics and Astronomy
GR II (Phys. 750) — Prof. Leo C. Stein — Spring 2020

**Problem Set 5 — SOLUTIONS**

**Due**: Thurs., Apr. 02, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. In class, we derived the formula for the leading gravitational radiation generated by a source,

$$h_{ij}^{\mathrm{TT}}(t,r) = \frac{2}{r}\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right)\ddot{I}_{kl}(t-r)\,, \tag{1}$$

where an overdot denotes derivative with respect to $u = t - r$, the second mass moment is $I_{kl} = \int \rho x^k x^l dV$, and the orthogonal projection tensor was $P_{ij} = \delta_{ij} - n_i n_j$ with $n^i = x^i/r$.

Show that we get the same result if we replace $I_{ij}$ with the tracefree mass quadrupole tensor, $\mathcal{I}_{kl} \equiv I_{kl} - \frac{1}{3}\delta_{kl}I$, where $I = \delta_{ij}I_{ij}$ is the trace.

Note that this is not trivial: there are two different types of trace removal. One is a three-dimensional trace, the other a 2-dimensional trace in the space orthogonal to $n^i$.

The significance here is that while $I$ has 6 components, $\mathcal{I}$ only has 5 independent components, which is the correct number for a radiative quadrupole (an $l$-pole should have $2l+1$ components).

***Solution:*** If these two ways of writing $h_{ij}^{\mathrm{TT}}$ are equivalent, then it is the projection of the trace-adjustment that must vanish:

$$\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right)\ddot{I}_{kl}(t-r) = \left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right)\ddot{\mathcal{I}}_{kl}(t-r) \tag{2}$$

$$\implies \qquad 0 = \left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right)\frac{1}{3}\delta_{kl}I\,. \tag{3}$$

Let's check if this is so, by using the properties of the projector tensors. The scalar factors $I/3$ are irrelevant.

$$\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right)\delta_{kl} = P_{ik}P_{jk} - \frac{1}{2}P_{ij}P_{kk} \tag{4}$$

$$= P_{ij} - \frac{1}{2}P_{ij}2 = 0\,, \tag{5}$$

since a projector squared is again a projector, and since the trace of the transverse projector is 2, the dimension of the image.

2. **Circular binary** Let's consider a circular binary with two point particle components of masses $m_1, m_2$ in a circular orbit lying in the $x-y$ plane. Let the separation be $R$, and to start we'll take the orbit to be Newtonian. Now let's compute the gravitational waves and the backreaction on the orbit. [Hint: everything will be simpler in terms of a reduced mass $\mu = m_1 m_2/(m_1+m_2)$ going around a central body of the total mass $M = m_1 + m_2$.]

   (a) Compute the gravitational wave tensor $h_{ij}^{\mathrm{TT}}$ at a point on the $z$ axis.

   ***Solution:*** We'll use the quadrupole formula, so we need the stress-energy tensor component $T^{00}$ to find the quadrupole moment. For slow velocities,

$$T^{00} = m_1\delta^3(\boldsymbol{x} - \boldsymbol{r}_1(t)) + m_2\delta^3(\boldsymbol{x} - \boldsymbol{r}_2(t))\,, \tag{6}$$

(we drop $\mathcal{O}(v^2)$ corrections due to relativistic effects). The quadrupole tensor is

$$I^{ij} = \int d^3x T^{00} x^i x^j = m_1 r_1^i r_1^j + m_2 r_2^i r_2^j \,. \tag{7}$$

For the trajectories we take just Newtonian circular orbits,

$$\vec{r}_1 = \frac{2R\mu}{m_1} (\cos(\Omega t), \sin(\Omega t), 0) \tag{8}$$

$$\vec{r}_2 = -\frac{2R\mu}{m_2} (\cos(\Omega t), \sin(\Omega t), 0) \ , \tag{9}$$

where $\Omega = \sqrt{GM/a^3} = \sqrt{GM/8R^3}$. Plugging this into the quadrupole tensor we get

$$\boldsymbol{I} = 4R^2\mu \begin{pmatrix} \cos^2(\Omega t) & \cos(\Omega t)\sin(\Omega t) & 0 \\ \cos(\Omega t)\sin(\Omega t) & \sin^2(\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10}$$

$$= 2R^2\mu \begin{pmatrix} 1+\cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & 1-\cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{11}$$

Let's take the second time derivative,

$$\ddot{\boldsymbol{I}} = -8\Omega^2 R^2 \mu \begin{pmatrix} \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & -\cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{12}$$

To evaluate the quadrupole formula on the $z$ axis, we will use projectors to make the tensor orthogonal to $z$ – but notice that it already is! Then there's a term to remove the trace – and again notice that this is already trace-free. So our result is

$$\bar{\boldsymbol{h}}^{TT}(t,0,0,z) = \boldsymbol{h}^{TT} = \frac{2G}{r}(PP - \frac{1}{2}PP)\ddot{I}(t_r) = \frac{2G}{r}\ddot{\boldsymbol{I}}(t_r) \tag{13}$$

$$= -\frac{16\Omega^2 R^2 \mu}{r} \begin{pmatrix} \cos(2\Omega t_r) & \sin(2\Omega t_r) & 0 \\ \sin(2\Omega t_r) & -\cos(2\Omega t_r) & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{14}$$

where the retarded time is $t_r = t - |\boldsymbol{x}| = t - z$.

(b) Compute the energy loss due to gravitational waves (integrating over all emission directions). Remember that this can only be interpreted correctly when averaged over a few cycles of the radiation.

***Solution:*** The easiest way to do this is with the leading power-loss formula in terms of the trace-free quadrupole tensor,

$$P_{GW} = -\frac{G}{5}\left\langle \dddot{I\!\!\!-}_{ij} \dddot{I\!\!\!-}^{ij} \right\rangle , \tag{15}$$

where as above, $I\!\!\!-_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I$. Notice that here, the trace $I = 4R^2\mu = \text{const.}$, so it vanishes when taking time derivatives. This need not be the case, we're lucky! Therefore we get to substitute $\dddot{I\!\!\!-}_{ij} = \dddot{I}_{ij}$. Calculating we get

$$P_{\text{GW}} = -\frac{512 G R^4 \mu^2 \Omega^6}{5} = -\frac{G^4 M^3 \mu^2}{5R^5} \,. \tag{16}$$

(c) Now claiming that energy is conserved,

$$\frac{d}{dt}(E_{\text{kin.}} + E_{\text{pot.}} + E_{\text{GW}}) = 0 \,, \tag{17}$$

derive an equation for how the orbit must shrink, $dR/dt$. [Hint: $E_{\text{kin.}} + E_{\text{pot.}}$ combine into a very simple expression of $R$ for a bound orbit.]

***Solution:*** Recall that for a bound Kepler orbit,

$$E_{\text{kin.}} + E_{\text{pot.}} = -\frac{GM\mu}{2a} = -\frac{GM\mu}{4R}\,. \tag{18}$$

So, using the preceding result, we get the ODE

$$-\frac{d}{dt}\frac{GM\mu}{4R} = -\frac{G^4M^3\mu^2}{5R^5} \tag{19}$$

$$\frac{GM\mu}{4}\frac{1}{R^2}\frac{dR}{dt} = -\frac{G^4M^3\mu^2}{5R^5} \tag{20}$$

$$\frac{dR}{dt} = -\frac{4G^3M^2\mu}{5R^3}\,. \tag{21}$$

(d) Derive the rate of change of orbital frequency $\Omega$ caused by emission of GWs. You should get something in terms of the chirp mass, $\mathcal{M} \equiv \mu^{3/5}M^{2/5}$, to some power.

***Solution:*** Since $\Omega(t) = \sqrt{GM/8}R(t)^{-3/2}$,

$$\frac{d\Omega}{dt} = -\frac{3}{2}\sqrt{\frac{GM}{8}}R(t)^{-5/2}\frac{dR}{dt} = \frac{6\mu}{5}\sqrt{\frac{G^7M^5}{8R^{11}}} \tag{22}$$

$$\frac{d\Omega}{dt} = \frac{96}{5}\left(G\mathcal{M}\Omega^{11/5}\right)^{5/3}\,. \tag{23}$$

(e) Integrate $d\Omega/dt$ to find the solution for $\Omega(t)$. This will diverge at some coalescence time $T_{\text{coal.}}$ (this is an artifact of the point particle treatment of the bodies). Your solution should be some power law for $(T_{\text{coal.}} - t)$.

***Solution:*** Rearranging and then integrating,

$$\frac{5}{96(G\mathcal{M})^{5/3}}\frac{d\Omega}{\Omega^{11/3}} = dt \tag{24}$$

$$\frac{5}{96(G\mathcal{M})^{5/3}}\frac{-3}{8}\Omega(t)^{-8/3} = t - T_{\text{coal.}}\,, \tag{25}$$

since we have the endpoint condition that $\Omega \to \infty$ as $t \to T_{\text{coal.}}$. Then solving we get

$$\Omega(t) = \left[\frac{5}{256(G\mathcal{M})^{5/3}}\frac{1}{T_{\text{coal.}} - t}\right]^{3/8}\,. \tag{26}$$

3. **Wave equation for Riemann**. While a metric perturbation is not gauge invariant, the linearized Riemann tensor is when we're on a flat background (do you remember why?). So, let's get a wave equation for the Riemann tensor itself.

(a) Starting from the Einstein equations and the full Bianchi identity,

$$\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0\,, \tag{27}$$

derive an equation for some appropriate divergence,

$$\nabla_a R^a{}_{bcd} = 8\pi G[\text{sources involving one derivative of } T]\,. \tag{28}$$

***Solution:*** Contract on two indices, the first coming from the set $a, b, c$ and the second coming from the set $d, e$. One term will be the divergence of Riemann, the other two will turn into Ricci, where you apply Einstein's equations. Thus find

$$\nabla_a R^a{}_{ebc} = 8\pi(\nabla_b \bar{T}_{ce} + \nabla_c \bar{T}_{be})\,. \tag{29}$$

(b) Now again starting from the Bianchi identity, derive an equation for

$$\Box R_{abcd} = 8\pi G[\text{sources built from two derivatives of } T\text{, and terms quadratic in } R]\,. \tag{30}$$

Here you will also make use of the result you got in item 3a. Note that there is a lot of room for error in the algebra for this problem. You may want to only work at the linearized level around a flat background, but I encourage you to work the full problem.

***Solution:*** Apply $\nabla^a$ to Eq. (27). One term forms the box operator acting on Riemann. In the other two, commute the outer derivative inwards so as to form divergence of Riemann. This commutation generates Riemann$^2$ terms (which vanish on a flat background). For the divergence terms, using the preceding result, which will now be in terms of a double-derivative of $T_{ab}$. See the MATHEMATICA notebook for the full solution.

(c) Now specialize to a plane wave on a flat background, so that the curvature wave has the form $R_{abcd} = R_{abcd}(t - z)$. Use the Bianchi identities and symmetries of Riemann to show that the only independent components are $R_{0i0j}$ (and others related by symmetries). [Hint: because we're on a flat background and linearizing, you can use a Fourier expansion and work mode by mode, using as an ansatz $R_{abcd} = C_{abcd}\exp(ik_e x^e)$ with a null wave-vector $k_e$ pointing in the $z$ direction, and some constant polarization tensor $C$].

***Solution:*** Every covariant derivative on $R_{abcd}$ will give a factor of $ik$, so the Bianchi identity becomes

$$k_a R_{bcde} + k_b R_{cade} + k_c R_{abde} = 0\,. \tag{31}$$

In flat space and rectangular coordinates, the $k$ vector will be $k^a \doteq (\omega, 0, 0, \omega)$. Examining the $a = 0$ component of Eq. (31), we can solve

$$R_{bcde} = \frac{1}{\omega}(k_b R_{c0de} + k_c R_{0bde}) = \frac{1}{\omega}(-k_b R_{0cde} + k_c R_{0bde})\,. \tag{32}$$

We can further expand the RHS by inserting this same identity. Suppose we tried to apply this to $R_{0cde}$. Then we would get $R_{0cde} = \frac{1}{\omega}[-(-\omega)R_{0cde} + k_c 0] = R_{0cde}$, no new information. Therefore first apply the exchange symmetry, $R_{0cde} = R_{de0c}$, and similarly for the second term in Eq. (32). Doing so we get

$$R_{bcde} = \frac{1}{\omega^2}(k_b k_d R_{0e0c} - k_b k_e R_{0d0c} - k_c k_d R_{0e0b} + k_c k_e R_{0d0b})\,. \tag{33}$$

Since Riemann is antisymmetric on adjacent pairs, something of the form $R_{0c0d}$ must have $c$ and $d$ spatial. So, every component of Riemann is in terms of $R_{0i0j}$.

(d) Show that in the above curvature wave propagating in the $z$ direction, the only nonvanishing components are $R_{0x0x} = -R_{0y0y}$, and $R_{0x0y}$.

***Solution:*** Evaluate Eq. (33) with $bcde = 0zde$, and you find

$$R_{0zde} = \frac{-1}{\omega}(k_d R_{0e0z} - k_e R_{0d0z})\,. \tag{34}$$

The RHS will vanish if either $d = 0$ or $e = 0$, therefore $R_{0z0i} = 0$. Next suppose $d, e = i, j$ are two different spatial directions, then you see that the RHS still vanishes. Thus all possibilities with $R_{0zde} = 0$.

The only remaining components to study are $R_{0x0x}, R_{0y0y}$, $R_{0x0y}$, and $R_{0y0x}$. Next look at 00 component of the Ricci tensor, which vanishes because we are in a vacuum spacetime,

$$0 = R_{00} = \eta^{ab} R_{0a0b} = R_{0x0x} + R_{0y0y} + R_{0z0z}\,. \tag{35}$$

Since $R_{0z0z} = 0$, we find that $R_{0x0x} = -R_{0y0y}$. From the exchange symmetry of Riemann, $R_{0x0y} = R_{0y0x}$.

4. Contracting the Bianchi identities leads to the fact that the Einstein tensor is divergence-free, $\nabla_a G^a{}_b = 0$. Use this to *show* that $G^0{}_\mu$ must have fewer time derivatives than $G^i{}_\mu$. Thus conclude that the components $G^0{}_0$ and $G^0{}_i$ will be "constraint equations," while $G^i{}_j$ will be evolution equations that contain two time derivatives.

   ***Solution:*** Expanding out the contracted Bianchi identity gives

$$\partial_0 G^0{}_\mu = -\partial_i G^i{}_\mu + (\Gamma G \text{ terms with appropriate indices}). \tag{36}$$

   Now this is an identity which is satisfied for *all* metrics, not ones satisfying certain differential equations (like the Einstein field equations). In order to be an identity, whatever derivatives are acting on the metric must be the same on both sides of the equation. Otherwise this would be a differential equation for the metric instead of an identity.

   Now, focus on the highest derivative terms on both sides, which must cancel each other. We know that the Einstein tensor involves a first derivative of the Christoffel symbols, which themselves contain first derivatives of the metric, so $G \sim \partial^2 g$.

   The highest possible time derivative on the right is if $G^i{}_\mu$ contains two time derivatives, then the right hand side will include $\partial_i \partial_0^2 g$. This must be exactly balanced by the highest time derivative on the LHS. Since the LHS already contains $\partial_0 G^0{}_\mu$, we can have at most $G^0{}_\mu \supset \partial_i \partial_0 g$.

   The components $G^0{}_0$ and $G^0{}_i$ are constraint equations if the variables for our system are $g$ and $K \sim \partial_0 g$. Then we would be able to rewrite $G^0{}_0$ and $G^0{}_i$ in terms of spatial derivatives acting on $g$ and $K$, and no time derivatives of either.