UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
GR II (Phys. 750) — Prof. Leo C. Stein — Spring 2020

Problem Set 4

Due: Friday, Mar. 06, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Gravitation of spin in the weak-field limit**. Recall that in Lorenz gauge, expanded about flat space, and for a *static* source, the linearized Einstein field equations turn into

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \qquad \Longrightarrow \nabla^2 \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}\,, \tag{1}$$

with a Euclidean spatial Laplace operator ∇^2. We modeled a static weak-field source with $T_{00} = \rho$ and all other components being small enough that they don't matter. This recovered $\nabla^2\Phi = 4\pi G\rho$ where Φ was the Newtonian potential which appeared in the metric as $\bar{h}_{00} = -4\Phi$, giving $h_{\mu\nu} = -2\Phi \mathrm{diag}(1, 1, 1, 1)$.

Now we want to consider a slow rotation of the source, and see how this enters the weak-field metric.

(a) Let the source be spherically symmetric with radius R, and of uniform density ρ. Suppose it is rotating rigidly about the $x^3 = z$ axis with constant angular velocity Ω. Work out the components of the stress-energy tensor $T_{\mu\nu}$ to first order in Ω (do this in a center-of-momentum frame, with Cartesian coordinates centered on the center of mass).

If you wanted to go to order Ω^2, which components of $T_{\mu\nu}$ would change? No need to compute the correction, just indicate which terms would be corrected.

(b) Now use the linearized Einstein equations to find the components h_{0x}, h_{0y}, h_{0z} (trace reversal does not touch the off-diagonal components, since the background metric $\eta_{\mu\nu}$ is diagonal).

Hints:

- Make use of the Green's function for the Laplace operator. That is, if $\nabla^2 Q = -4\pi S$ for some field Q and source term S, then the formal solution for Q is

$$Q(x) = \int \frac{S(x')}{|x - x'|}\, d^3x'\,. \tag{2}$$

- The start of the expansion for $1/|x - x'|$, in tensor notation, is

$$\frac{1}{|x - x'|} = \frac{1}{r} + \frac{x^j x^{j\prime}}{r^3} + \cdots \tag{3}$$

(the summation is implied, we're allowed to be careless when the spatial meteric is δ_{ij}).

- Though you have to first set up the integrals in rectangular coordinates, it is easier to perform the integrals by transforming the $x^{j\prime}$ coordinates into spherical coordinates (r', θ', ϕ').

(c) Let's now use the Newtonian relationship between the spin angular momentum S, the moment of inertia I, and the angular velocity, $S^k = I\Omega^k$ (since the body is spherical, we only have the isotropic moment of inertia I instead of the whole tensor). Rewrite your result for h_{0i} in terms of S^k. [Note: Look at MTW Sec. 19.1 to see how to generalize this away from spherical symmetry and uniform density].

(d) Transform your rectangular-coordinate result for h_{0i} into a spherical coordinate system, giving $h_{0r}, h_{0\theta}, h_{0\phi}$ [hint: you should find that only one of these components is non-zero, and it should be proportional to $S^z \sin^2\theta/r$].

2. Let's take a weak-field metric that has a potential and a "vector" part,

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2 + dy^2 + dz^2) - 2\beta_i dx^i dt\,. \tag{4}$$

(a) Let's rewrite the geodesic equation for a particle with a slow velocity in 3-dimensional language. Working to first order in $\boldsymbol{v}$, show that the geodesic equation gives

$$m\frac{d^2\boldsymbol{x}}{dt^2} = m\boldsymbol{g} + m\boldsymbol{v}\times\boldsymbol{H}\,, \tag{5}$$

where $\boldsymbol{x}$ is the 3-position, $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$, and $\boldsymbol{H} = \boldsymbol{\nabla}\times\boldsymbol{\beta}$. Here all bold symbols are 3-dimensional.

(b) For stationary sources (i.e. the stress-energy tensor does not change with time), show that the Einstein equations are

$$\boldsymbol{\nabla}\cdot\boldsymbol{g} = -4\pi G\rho \tag{6}$$

$$\boldsymbol{\nabla}\times\boldsymbol{H} = -16\pi G\boldsymbol{J} \tag{7}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{H} = 0 \tag{8}$$

$$\boldsymbol{\nabla}\times\boldsymbol{g} = 0\,. \tag{9}$$

Here $\boldsymbol{J} \equiv \rho\boldsymbol{v}$ is the mass current of the fluid source. This is another place where you might want to use computer algebra to help.

3. Show that the Lorenz gauge condition $\nabla^\mu \bar{h}_{\mu\nu}$ is equivalent to the "harmonic" coordinate gauge condition, $\Box x^{(\mu)} = 0$. Remember that the μ on $x^{(\mu)}$ is not a vector index, but rather a label to count the coordinates.