UNIVERSITY OF MISSISSIPPI
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GR II (Phys. 750) — Prof. Leo C. Stein — Spring 2020

Problem Set 4 — SOLUTIONS

Due: Friday, Mar. 06, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Gravitation of spin in the weak-field limit**. Recall that in Lorenz gauge, expanded about flat space, and for a *static* source, the linearized Einstein field equations turn into

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \qquad \Longrightarrow \nabla^2\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}\,, \tag{1}$$

with a Euclidean spatial Laplace operator ∇^2. We modeled a static weak-field source with $T_{00} = \rho$ and all other components being small enough that they don't matter. This recovered $\nabla^2\Phi = 4\pi G\rho$ where Φ was the Newtonian potential which appeared in the metric as $\bar{h}_{00} = -4\Phi$, giving $h_{\mu\nu} = -2\Phi\mathrm{diag}(1, 1, 1, 1)$.

Now we want to consider a slow rotation of the source, and see how this enters the weak-field metric.

(a) Let the source be spherically symmetric with radius R, and of uniform density ρ. Suppose it is rotating rigidly about the $x^3 = z$ axis with constant angular velocity Ω. Work out the components of the stress-energy tensor $T_{\mu\nu}$ to first order in Ω (do this in a center-of-momentum frame, with Cartesian coordinates centered on the center of mass).

If you wanted to go to order Ω^2, which components of $T_{\mu\nu}$ would change? No need to compute the correction, just indicate which terms would be corrected.

Solution: In flat space, the velocity field for a body rigidly rotating about the Ω^i vector is $v^i = \epsilon^{ijk}\Omega^j x^k$, and we would have $v_i = v^i$. So, for rotation about the z axis, we would have $v^x = y\Omega$, $v^y = -x\Omega$, and $v^z = 0$. For non-relativistic velocities we would have $v^0 = 1$. In the perfect fluid stress-energy tensor, we get

$$T_{00} = \rho + \mathcal{O}(\Omega^2)\,, \tag{2}$$
$$T_{0i} = \rho v_i + \mathcal{O}(\Omega^3) = \rho v^i + \mathcal{O}(\Omega^3)\,, \tag{3}$$
$$T_{ij} = \mathcal{O}(\Omega^2)\,. \tag{4}$$

The expansion in small Ω is also a slow-velocity expansion. The velocity gets a relativistic correction that is at the *relative* $x^2\Omega^2/c^2$ order, hence the correction terms above.

(b) Now use the linearized Einstein equations to find the components h_{0x}, h_{0y}, h_{0z} (trace reversal does not touch the off-diagonal components, since the background metric $\eta_{\mu\nu}$ is diagonal).

Hints:

- Make use of the Green's function for the Laplace operator. That is, if $\nabla^2 Q = -4\pi S$ for some field Q and source term S, then the formal solution for Q is

$$Q(x) = \int \frac{S(x')}{|x - x'|}\, d^3x'\,. \tag{5}$$

- The start of the expansion for $1/|x - x'|$, in tensor notation, is

$$\frac{1}{|x - x'|} = \frac{1}{r} + \frac{x^j x^{j\prime}}{r^3} + \cdots \tag{6}$$

(the summation is implied, we're allowed to be careless when the spatial meteric is δ_{ij}).

- Though you have to first set up the integrals in rectangular coordinates, it is easier to perform the integrals by transforming the $x^{j\prime}$ coordinates into spherical coordinates (r', θ', ϕ').

Solution: Using the expansion and the tips, let's compute:

$$h_{0i}(x) = \bar{h}_{0i}(x) = 4G \int \frac{T_{0i}(x')}{|x - x'|} d^3x' = \frac{4G\rho}{r} \left[\int v^i(x') d^3x' + \frac{x^l}{r^2} \int x'^l v^i(x') d^3x' + \dots \right] . \tag{7}$$

Now if we replace $v^i(x') = \epsilon^{ijk}\Omega^j x'^k$ we see that the first term vanishes, since it is odd in x'. Continuing, we need to evaluate

$$h_{0i}(x) = \frac{4G\rho}{r^3} \epsilon^{ijk}\Omega^j x^l \int x'^l x'^k d^3x' = \frac{4G\rho}{r^3} \epsilon^{ijk}\Omega^j x^l \int \hat{n}'^l \hat{n}'^k r'^4 dr' d^2\Omega' , \tag{8}$$

where we have used $r'^i = r'\hat{n}'^i$ and $d^2\Omega$ is the area element on the unit 2-sphere. Now we can do the radial integral right away, as the source only extends from $r' = 0$ to $r' = R$. Finally we will need the identity

$$\int \hat{n}^i \hat{n}^j d^2\Omega = \frac{4\pi}{3} \delta^{ij} , \tag{9}$$

which can be shown explicitly by integrating in spherical coordinates. Assembling we find

$$h_{0i}(x) = \frac{16\pi G\rho R^5}{15 r^3} \epsilon^{ijk}\Omega^j x^k , \tag{10}$$

or $h_{0x} = 16\pi G\rho R^5 \Omega y/15r^3$, $h_{0y} = -16\pi G\rho R^5 \Omega x/15r^3$, and $h_{0z} = 0$.

(c) Let's now use the Newtonian relationship between the spin angular momentum S, the moment of inertia I, and the angular velocity, $S^k = I\Omega^k$ (since the body is spherical, we only have the isotropic moment of inertia I instead of the whole tensor). Rewrite your result for h_{0i} in terms of S^k. [Note: Look at MTW Sec. 19.1 to see how to generalize this away from spherical symmetry and uniform density].

Solution: For a sphere of uniform density, the total mass is $M = \frac{4\pi}{3} R^3 \rho$ and the moment of inertia is $I = \frac{2}{5} M R^2 = \frac{8\pi}{15} R^5 \rho$. Substituting in we get

$$h_{0i}(x) = \frac{2G}{r^3} \epsilon^{ijk} S^j x^k . \tag{11}$$

(d) Transform your rectangular-coordinate result for h_{0i} into a spherical coordinate system, giving $h_{0r}, h_{0\theta}, h_{0\phi}$ [hint: you should find that only one of these components is non-zero, and it should be proportional to $S^z \sin^2\theta/r$].

Solution: We want to go to the barred spherical coordinates $x^{\bar{i}} = \{r, \theta, \phi\}$ in terms of the unprimed Cartesian coordinates $x^i = \{x, y, z\}$, using the standard transformation. Thus we will have to compute the Jacobian matrix in

$$h_{0\bar{j}} = \frac{\partial x^i}{\partial x^{\bar{j}}} h_{0i} . \tag{12}$$

Since this is a second rank tensor, there are actually two Jacobian factors, but this coordinate transformation is diagonal and purely spatial – thus h_{0i} ends up transforming like a spatial covector. Computing the Jacobian we get

$$\begin{pmatrix} h_{0r} \\ h_{0\theta} \\ h_{0\phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix} \begin{pmatrix} h_{0x} \\ h_{0y} \\ h_{0z} \end{pmatrix} . \tag{13}$$

Inserting the Cartesian components, we get $h_{0r} = h_{0\theta} = 0$, and $h_{0\phi} = -2GS^z \sin^2\theta/r$.

2. Let's take a weak-field stationary metric that has a potential and a "vector" part,

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2) + 2\beta_i dx^i dt\,. \tag{14}$$

(a) Let's rewrite the geodesic equation for a particle with a slow velocity in 3-dimensional language. Working to first order in $\boldsymbol{v}$, show that the geodesic equation gives

$$m\frac{d^2\boldsymbol{x}}{dt^2} = m\boldsymbol{g} + m\boldsymbol{v}\times\boldsymbol{H}\,, \tag{15}$$

where $\boldsymbol{x}$ is the 3-position, $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$, and $\boldsymbol{H} = \boldsymbol{\nabla}\times\boldsymbol{\beta}$. Here all bold symbols are 3-dimensional.

Solution: The full geodesic equation is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0\,. \tag{16}$$

Not only do we need to expand the metric in a small parameter ϵ_1, where Φ and β are both $\mathcal{O}(\epsilon_1)$, but we will also expand the particle's velocity in *another* small parameter ϵ_2, where $v^i = \mathcal{O}(\epsilon_2)$, and $v^0 = 1 + \mathcal{O}(\epsilon_2^2)$. Then for the spatial components of the geodesic equation, we will need $\Gamma^i{}_{00}$ and $\Gamma^i{}_{0j}$, but not Γ^i_{jk}.

Now working to first order in ϵ_1, we get

$$\Gamma^i{}_{00} = \frac{1}{2}g^{i\mu}\left(2\partial_0 g_{0\mu} - \partial_\mu g_{00}\right) = -\frac{1}{2}\delta^{ij}\partial_j h_{00} = +\partial_i\Phi\,, \tag{17}$$

$$\Gamma^i{}_{0j} = \frac{1}{2}g^{i\mu}\left(\partial_0 g_{j\mu} + \partial_j g_{0\mu} - \partial_\mu g_{0j}\right) = \frac{1}{2}\delta^{ik}(\partial_j h_{0k} - \partial_k h_{0j}) = \frac{1}{2}(\partial_j\beta_i - \partial_i\beta_j)\,. \tag{18}$$

Now plugging this back into the geodesic equation, to first order in both ϵ_1 and ϵ_2, and using that $d\tau = dt(1+\mathcal{O}(\epsilon_2^2))$,

$$\frac{d^2x^i}{dt^2} = -\Gamma^i{}_{00}v^0v^0 - 2\Gamma^i{}_{0j}v^0v^j = -\partial_i\Phi + v^j(\partial_i\beta_j - \partial_j\beta_i)\,. \tag{19}$$

This last expression can be rewritten in traditional grad and curl notation, using that $(\boldsymbol{\nabla}\times\boldsymbol{\beta})^k = \epsilon^{klm}\partial_l\beta_m$, $(\boldsymbol{v}\times\boldsymbol{H})^i = \epsilon^{ijk}v^jH^k$, and using the epsilon identity $\epsilon^{ijk}\epsilon^{klm} = \delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}$.

(b) For stationary sources (i.e. the stress-energy tensor does not change with time), show that the Einstein equations are

$$\boldsymbol{\nabla}\cdot\boldsymbol{g} = -4\pi G\rho \tag{20}$$

$$\boldsymbol{\nabla}\times\boldsymbol{H} = -16\pi G\boldsymbol{J} \tag{21}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{H} = 0 \tag{22}$$

$$\boldsymbol{\nabla}\times\boldsymbol{g} = 0\,. \tag{23}$$

Here $\boldsymbol{J} \equiv \rho\boldsymbol{v}$ is the mass current of the fluid source. This is another place where you might want to use computer algebra to help.

Solution: The latter two equations are identities from the definitions of $\boldsymbol{g}$ as a gradient, and $\boldsymbol{H}$ as a curl.

The linearized Einstein tensor on a flat background is gauge-invariant – we can use the general expression (from e.g. Flanagan and Hughes (2005), or from Chapter 7 of Carroll) and apply it to whatever gauge we have here. In the notation of Carroll, we have $\Psi = \Phi$, $w_j = \beta_j$, and $s_{ij} = 0$. Then the two relevant components of the Einstein tensor are

$$G_{00} = 2\nabla^2\Psi + \partial_k\partial_l s^{kl} \tag{24}$$

$$G_{0j} = -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s_j{}^k\,. \tag{25}$$

Putting this into the EFE in our notation, for a stationary metric, gives

$$8\pi G\rho = -2\boldsymbol{\nabla}\cdot\boldsymbol{g}\,, \tag{26}$$

$$8\pi G J_i = -\frac{1}{2}\nabla^2\beta_j + \frac{1}{2}\partial_j\partial_k\beta^k\,. \tag{27}$$

Finally use the identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{\beta}) = -\nabla^2\boldsymbol{\beta} + \nabla(\boldsymbol{\nabla}\cdot\boldsymbol{\beta})$, which can also be proved with the identity for the contraction of two epsilon tensors.

3. Show that the Lorenz gauge condition $\nabla^\mu\bar{h}_{\mu\nu} = 0$ is equivalent to the "harmonic" coordinate gauge condition, $\Box x^{(\mu)} = 0$. Remember that the μ on $x^{(\mu)}$ is not a vector index, but rather a label to count the coordinates.

   Solution: Starting from the harmonic gauge condition, compute:

$$0 = g^{\rho\sigma}\nabla_\rho\nabla_\sigma x^{(\mu)} = g^{\rho\sigma}\nabla_\rho\partial_\sigma x^{(\mu)} = g^{\rho\sigma}\nabla_\rho\delta^{(\mu)}_\sigma \tag{28}$$

$$= g^{\rho\sigma}\left(\partial_\rho\delta^{(\mu)}_\sigma - \Gamma^\nu{}_{\rho\sigma}\delta^{(\mu)}_\nu\right) \tag{29}$$

$$= -g^{\rho\sigma}\Gamma^\mu{}_{\rho\sigma}\,. \tag{30}$$

   Here only the lower index on $\delta^{(\mu)}_\sigma$ is a tensor index that needed to be corrected. Now suppose we perturb the metric, $g_{\mu\nu} \to g_{\mu\nu} + \epsilon h_{\mu\nu} + \mathcal{O}(\epsilon^2)$, and similarly perturb the gauge condition. Then the perturbation of the condition is:

$$0 = \delta(g^{\rho\sigma}\Gamma^\mu{}_{\rho\sigma}) = \delta(g^{\rho\sigma})\Gamma^\mu{}_{\rho\sigma} + g^{\rho\sigma}\delta\Gamma^\mu{}_{\rho\sigma}\,. \tag{31}$$

   Now recall that $\delta(g^{\rho\sigma}) = -h^{\rho\sigma}$, and

$$\delta\Gamma^\mu{}_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\left(\nabla_\rho h_{\sigma\nu} + \nabla_\sigma h_{\rho\nu} - \nabla_\nu h_{\rho\sigma}\right)\,. \tag{32}$$

   Contracting we get

$$g^{\rho\sigma}\delta\Gamma^\mu{}_{\rho\sigma} = g^{\mu\nu}\left(\nabla_\rho h^\rho{}_\nu - \frac{1}{2}\nabla_\nu h\right) = \nabla_\rho\bar{h}^{\rho\mu}\,. \tag{33}$$

   Now, unfortunately, the problem statement was a little vague. The above identity along with Eq. (31) only give the Lorenz gauge condition if we satisfy $h^{\rho\sigma}\Gamma^\mu{}_{\rho\sigma} = 0$, for example if the background is flat with rectangular coordinates.