UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
GR II (Phys. 750) — Prof. Leo C. Stein — Spring 2020

Problem Set 2

Due: Friday, Feb. 14, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. Suppose we have an algebra $\mathcal{A}$, and any two derivations on that algebra, D_1 and D_2 (recall that a derivation satisfies the Leibniz rule, $D(ab) = D(a)b + aD(b)$). Show that the commutator $[D_1, D_2](a) = D_1 D_2 a - D_2 D_1 a$ is also a derivation.

2. Show that every three vector fields $a, b, c \in \mathfrak{X}(\mathcal{M})$ on a manifold $\mathcal{M}$ satisfy the Jacobi identity,

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0\,. \tag{1}$$

3. Suppose we have a vector bundle E over the base manifold $\mathcal{M}$, and we have a connection (or covariant derivative) D such that the operation $D_v : \Gamma(E) \to \Gamma(E)$ satisfies:

$$D_v(fs + t) = v(f)s + fD_v(s) + D_v(t) \tag{2}$$
$$D_{fv+w}(s) = fD_v(s) + D_w(t)\,, \tag{3}$$

for scalar function $f \in C^\infty(\mathcal{M})$, vector fields $v, w \in \mathfrak{X}(\mathcal{M})$, and sections $s, t \in \Gamma(E)$.

Show the following:

(a) If you have this connection D and another connection D^0 (also satisfying these rules), that the difference $D - D^0$ is a tensor, in the sense that it does not take a derivative of its argument:

$$D_v(fs) - D^0_v(fs) = f\left(D_v(s) - D^0_v(s)\right)\,. \tag{4}$$

The fact that it does not take a derivative of v should be clear from the properties of a connection.

(b) We define the operation

$$F(v, w)s = D_v D_w s - D_w D_v s - D_{[v,w]} s\,. \tag{5}$$

Now it is not clear if v, w, or s do or don't get differentiated! Show that $F(v, w)$ is a tensor in the sense that it does not take a derivative of v, w, or s.

You probably remember the Bianchi identity for the Riemann tensor (curvature tensor on the tangent bundle TM),

$$\nabla_{[a} R_{bc]de} = 0\,. \tag{6}$$

It turns out that this is true for the connection on any vector bundle,

$$[D_u, [D_v, D_w]] + [D_v, [D_w, D_u]] + [D_w, [D_u, D_v]] = 0\,. \tag{7}$$

However I am not going to ask you to prove this. If you want to see what it takes, I refer you to page 253 of Baez and Muniain's *Gauge theories, knots, and gravity.*

4. Now let's focus on connections on the tangent bundle. Recall that we saw a coordinate calculation of the Lie derivative using a coordinate system's partial derivatives (a valid connection on the tangent bundle). That formula was

$$\mathcal{L}_v T^{i\cdots}{}_{j\dots} = v^k \partial_k T^{i\cdots}{}_{j\dots} - T^{k\cdots}{}_{j\dots}\partial_k v^i - \dots + T^{i\cdots}{}_{k\dots}\partial_j v^k + \dots \tag{8}$$

where there is a correction term with a minus sign for every upstairs index, and one with a minus sign for every downstairs index. Now suppose we have another connection on the tangent bundle, D, which is a *symmetric* connection (but we don't necessarily have a metric). Prove that you can use D instead of ∂ in Eq. (8) and get the same result.

5. Let's apply Frobenius' theorem to the following nonlinear system of PDEs:

$$\partial_x f_1 = A_{11}(x, y, f_1(x, y), f_2(x, y)) \tag{9a}$$
$$\partial_y f_1 = A_{12}(x, y, f_1(x, y), f_2(x, y)) \tag{9b}$$
$$\partial_x f_2 = A_{21}(x, y, f_1(x, y), f_2(x, y)) \tag{9c}$$
$$\partial_y f_2 = A_{22}(x, y, f_1(x, y), f_2(x, y))\,. \tag{9d}$$

We want to know what are necessary and sufficient conditions on the A functions for solutions to exist.

(a) To turn this into a geometry problem, we'll want to look for a submanifold in some bigger space (some bundle over $\mathbb{R}^2 \ni (x, y)$). Explain what is this bundle and what are local coordinates for it (thereby stating the dimension of the bundle). What is the dimension of the submanifold we're looking for?

(b) Turn the system (9) into a set of vector fields $X_{(i)}$ which define a distribution.

(c) Using Frobenius' theorem, compute the "integrability conditions", i.e. the necessary and sufficient conditions for existence of solutions, that the A's have to satisfy.