UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
GR II (Phys. 750) — Prof. Leo C. Stein — Spring 2020

Problem Set 1

Due: Tuesday, Feb. 4, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. Construct explicit examples (giving coordinate maps) of the following types of maps between manifolds:
   (a) injective but not bijective
   (b) surjective but not bijective
   (c) neither surjective nor injective; and
   (d) bijective (but not the identity map, that's too easy!)

   For each of these cases, pick for the domain and codomain any well-known manifold such as the real line $\mathbb{R}$ or space $\mathbb{R}^n$, an interval $[0, 1]$, an n-sphere S^n, Cartesian products of these, etc. (If you are using a non-standard coordinate system then explain the coordinates.)

2. Take the map (say it was called $F : M \to N$) you defined in item 1a and compute the differential dF in the coordinates you used above. Use this differential to compute the pullback of any one-form from N back to M.

3. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a logarithmic spiral, so that in rectangular coordinates we have $\gamma(t) = (e^t \cos t, e^t \sin t)$.
   (a) What is the pullback $\gamma^* r$ of the function $r = \sqrt{x^2 + y^2}$?
   (b) Find the matrix representing the differential $d\gamma$ in these coordinates.
   (c) Use $d\gamma$ to find the tangent vector to the curve.
   (d) Use $d\gamma$ to pull back the one-form dr.
   (e) Use $d\gamma$ to pull back the two-form $dx \wedge dy$.