UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2020

Problem Set 10 — SOLUTIONS

Due: Friday, May 8, 2020, by 5PM

Material: The final covers all the material so far.

Due date: Friday, May 8, 2020 by 5PM to 205 Lewis Hall. If my door is closed, please slide the exam under my door. Late exams will require extenuating circumstances.

Logistics: The exam consists of this page plus 2 pages of questions. Do not look at the problems until you are ready to start it.

Time: The work might expand to eat up as much time as you allot – therefore I highly recommend you restrict yourself to no more than 5 hours cumulative time spent on these problems. You may take as many breaks as you like, not counted against the 5 hours. **You should not be consulting references, working on the problems, or discussing with others during the breaks**.

Resources: The final is **not collaborative**. All questions must be done on your own, without consulting anyone else. You may consult your own notes (both in-class and notes on this class you or a colleague in the class have made), the textbook by Griffiths, and solution sets on the course website. **You may not consult any other material**, including other textbooks, the web (except for the current Phys. 402 website), material from previous years' Phys. 402 or any other classes, or copies you have made of such material, or any other resources. Calculators and symbolic manipulation programs are not allowed.

1. **Gradually changing waveguide**. Suppose we have a hollow waveguide running in the z direction for several kilometers. At every z the cross-section is a rectangle with sides a, b where $a \geq b$. Suppose we send in a wave that excites the TE_{mn} mode with some frequency ω and wavenumber k that satisfy the dispersion relation [Griffiths Eq. (9.187)]

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]} = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}\,, \tag{1}$$

and the amplitude is given by some B_0 in [Griffiths Eq. (9.186)]

$$B_z = B_0 \cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})\,. \tag{2}$$

(a) What is the average energy flux $\langle \boldsymbol{S} \rangle$ (averaged over one period of the wave)? What is the averaged energy flux through the whole cross-section, $\int \langle \boldsymbol{S} \rangle \cdot d\boldsymbol{a}$? (The result of Griffiths' problem 9.11 [time-averaging a product using complex exponentials] might be helpful).

Solution: We want to find $\langle \boldsymbol{S} \rangle = \frac{1}{2\mu_0}\tilde{\boldsymbol{E}} \times \tilde{\boldsymbol{B}}*$. The complex fields are $\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(kz-\omega t)}$ and $\tilde{\boldsymbol{B}}^* = \tilde{\boldsymbol{B}}_0^* e^{-i(kz-\omega t)}$. For the TE_{mn} mode, $E_z = 0$, and the entire solution is determined in terms of B_z. We find $E_{x,y}$ and $B_{x,y}$ from various derivatives of B_z [Eq. (9.180)], finding

$$B_x^* = \frac{-ik}{(\omega/c)^2 - k^2}(\frac{-m\pi}{a})B_0 \sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b}) \tag{3}$$

$$B_y^* = \frac{-ik}{(\omega/c)^2 - k^2}(\frac{-n\pi}{b})B_0 \cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b}) \tag{4}$$

$$E_x = \frac{-i\omega}{(\omega/c)^2 - k^2}(\frac{-n\pi}{b})B_0 \cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b}) \tag{5}$$

$$E_y = \frac{-i\omega}{(\omega/c)^2 - k^2}(\frac{-m\pi}{a})B_0 \sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})\,. \tag{6}$$

Now computing the cross product and averaging, we get

$$\begin{aligned}\langle \boldsymbol{S} \rangle = \frac{1}{2\mu_0}\frac{B_0^2}{(\omega/c)^2 - k^2}\Bigg\{ & \frac{i\pi\omega m}{a}\sin(\frac{m\pi x}{a})\cos(\frac{m\pi x}{a})\cos^2(\frac{n\pi y}{b})\hat{\boldsymbol{x}} \\ & + \frac{i\pi\omega n}{b}\cos^2(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b})\cos(\frac{n\pi y}{b})\hat{\boldsymbol{y}} \\ & + \frac{\omega k\pi^2}{(\omega/c)^2 - k^2}\left[(\frac{n}{b})^2\cos^2(\frac{m\pi x}{a})\sin^2(\frac{n\pi y}{b}) + (\frac{m}{a})^2\sin^2(\frac{m\pi x}{a})\cos^2(\frac{n\pi y}{b})\right]\hat{\boldsymbol{z}}\Bigg\}\,.\end{aligned} \tag{7}$$

Finally we integrate over x from 0 to a, and over y from 0 to b, to find

$$\int \langle \boldsymbol{S} \rangle \cdot d\boldsymbol{a} = \frac{1}{8\mu_0}\frac{\omega k\pi^2 B_0^2}{[(\omega/c)^2 - k^2]^2}ab\left[(\frac{m}{a})^2 + (\frac{n}{b})^2\right] = \frac{\omega k a b c^2}{8\mu_0\omega_{mn}^2}B_0^2\,. \tag{8}$$

Now suppose that this waveguide's cross-section changes very slowly in z, so that $a = a(z)$ and $b = b(z)$. Very slowly here means that $\frac{1}{a}\frac{da}{dz} \ll k$ and similarly for b. For simplicity we will assume that the aspect ratio a/b remains constant.

(b) Will ω change with z? What about k?

Solution: The frequency ω will not change, but k will.

(c) Now the amplitude $B_0(z)$ will have to slowly vary with z. Find a differential equation that would allow you to solve for $B_0(z)$ if somebody gave you $a(z)$ (and thus they are also giving you $b(z)$ since their ratio is constant).

Solution: The energy flux through each z must be the same for energy to be conserved. Therefore we must have

$$\frac{d}{dz}\int \langle \boldsymbol{S} \rangle \cdot d\boldsymbol{a} = 0\,. \tag{9}$$

Therefore we have

$$0 = \frac{d}{dz}\left[\frac{\omega k a b c^2}{8\mu_0\omega_{mn}^2}B_0^2\right]. \tag{10}$$

Inside the brackets, a, b, k, ω_{mn}, and B_0 are all functions of z.

(d) Find a simple combination of a, B_0, and k that is constant along z.

Solution: Start from Eq. (10). Then replace b with $b = \epsilon a$ where $\epsilon = b/a$ is a constant. We can ignore several constants that can be taken out of the derivative and divided out. Now we have

$$0 = \frac{d}{dz}\left[\frac{ka^2}{(m/a)^2 + (n/\epsilon a)^2}B_0^2\right] \tag{11}$$

$$0 = \frac{d}{dz}\left[\frac{ka^4}{m^2 + (n/\epsilon)^2}B_0^2\right]. \tag{12}$$

The denominator is a constant, so we have found $ka^4B_0^2$ is a constant along z.

(e) What will happen if a gradually shrinks too small?

Solution: If a becomes too small, ω_{mn} will grow to be larger than ω, and the mode will not be able to propagate any more. It will reflect back down the waveguide in the opposite direction. Notice that as ω_{mn} approaches ω, $k \to 0$, so it is impossible to shrink a to this point adiabatically (satisfying $\frac{1}{a}\frac{da}{dz} \ll k$).

2. **Integral identities**. For the following problems, you can assume that as you go to very large distances, the electric field decays as $1/r^2$, and the magnetic field decays as $1/r^3$.

(a) How quickly can the vector potential $\boldsymbol{A}$ decay at large r?

Solution: It can decay as $1/r^2$; one derivative of $\boldsymbol{A}$ gives $\boldsymbol{B}$.

(b) Prove the following integral identity, for any two vector fields $\boldsymbol{V}, \boldsymbol{W}$ integrated over volume $\mathcal{V}$:

$$\int_{\mathcal{V}} \boldsymbol{W}\cdot(\nabla\times\boldsymbol{V})\,d^3r = \int_{\partial\mathcal{V}}(\boldsymbol{V}\times\boldsymbol{W})\cdot d\boldsymbol{a} + \int_{\mathcal{V}}\boldsymbol{V}\cdot(\nabla\times\boldsymbol{W})\,d^3r\,. \tag{13}$$

Solution: This comes simply from rearranging the product rule

$$\nabla\cdot(\boldsymbol{V}\times\boldsymbol{W}) = \boldsymbol{W}\cdot(\nabla\times\boldsymbol{V}) - \boldsymbol{V}\cdot(\nabla\times\boldsymbol{W})\,, \tag{14}$$

then integrating over $\mathcal{V}$, and applying the divergence theorem on the appropriate term.

(c) Now combine everything to show that the following integral over all space and time vanishes:

$$\int_{-\infty}^{+\infty}\int_{\text{All space}}(\boldsymbol{E}\cdot\boldsymbol{B})\,d^3r\,dt = 0\,. \tag{15}$$

[Hint: use the potential formulation, and assume that the fields also vanish as $t \to \pm\infty$.]

Solution: We have $\boldsymbol{E} = -\nabla V - \frac{\partial}{\partial t}\boldsymbol{A}$ and $\boldsymbol{B} = \nabla\times\boldsymbol{A}$. Our integral is now

$$\int_{-\infty}^{+\infty}\int_{\text{All space}}\left[-\nabla V\cdot(\nabla\times\boldsymbol{A}) - \frac{\partial\boldsymbol{A}}{\partial t}\cdot(\nabla\times\boldsymbol{A})\right]d^3r\,dt\,. \tag{16}$$

In the first term, integrated by parts to move the gradient from V onto $\boldsymbol{B}$. The new integrand vanished because $\boldsymbol{B}$ is divergence-free. The boundary term vanishes because the fields decay sufficiently rapidly.

Now we have to handle

$$\int_{-\infty}^{+\infty}\int_{\text{All space}}\left[-\frac{\partial\boldsymbol{A}}{\partial t}\cdot(\nabla\times\boldsymbol{A})\right]d^3r\,dt\,. \tag{17}$$

Using the identity in Eq. (13), we can rewrite this as

$$\int_{-\infty}^{+\infty} \int_{\text{All space}} [-\frac{\partial \boldsymbol{A}}{\partial t} \cdot (\nabla \times \boldsymbol{A})] \, d^3r \, dt =$$
$$\int_{-\infty}^{+\infty} \int_{\text{Boundary}} [-\frac{\partial \boldsymbol{A}}{\partial t} \times \boldsymbol{A}] \cdot d\boldsymbol{a} \, dt + \int_{-\infty}^{+\infty} \int_{\text{All space}} [-\boldsymbol{A} \cdot (\nabla \times \frac{\partial \boldsymbol{A}}{\partial t})] \, d^3r \, dt$$

In the second integral on the RHS, the time derivative commutes with the curl. Then integrate by parts the time derivative to put it on the first factor. Now this term is the same as the one on the LHS, except for a sign; they combine. Now we have

$$\int_{-\infty}^{+\infty} \int_{\text{All space}} [\frac{\partial \boldsymbol{A}}{\partial t} \cdot (\nabla \times \boldsymbol{A})] \, d^3r \, dt = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{\text{Boundary}} [\frac{\partial \boldsymbol{A}}{\partial t} \times \boldsymbol{A}] \cdot d\boldsymbol{a} \, dt \tag{18}$$

and we want to show that this vanishes. But the term on the right hand side is integrated over an arbitrarily large 2-sphere with radius R. Each factor of $\boldsymbol{A}$ inside decays as $1/R^2$, and the area grows as R^2, so the entire integral decays as $1/R^2$ overall. Thus it vanishes as $R \to \infty$.

3. **Radiation reaction from a rotor with two charges**. One problem set 8, we considered a rotor of length $2b$ laying in the $x - y$ plane, with a charge $+q$ at one end and $-q$ at the other end, spinning about the z axis at angular frequency ω. In that problem, we found the dipole radiation carried an energy flux (integrated over all angles):

$$\langle \frac{dE}{dt} \rangle = \frac{2\mu_0 q^2 b^2 \omega^4}{3\pi c} \, . \tag{19}$$

Let us now give each of these charges a mass m, and let the rotor not contribute to the moment of inertia.

(a) Since energy is leaving, the spin rate $\omega(t)$ will slowly decrease. Find a differential equation for $d\omega/dt$ and solve for $\omega(t)$.

Solution: Energy conservation. All the energy is in rotational kinetic energy, $E = \frac{1}{2} I \omega^2$, with $I = 2mb^2$. Combining we get

$$\frac{1}{2}(2mb^2) 2\omega \frac{d\omega}{dt} = -\frac{2\mu_0 q^2 b^2 \omega^4}{3\pi c} \, . \tag{20}$$

Solving,

$$\frac{d\omega}{dt} = -\frac{\mu_0 q^2 \omega^3}{3\pi m c} \equiv -\alpha \omega^3 \, , \tag{21}$$

where $\alpha \equiv \mu_0 q^2 / 3\pi m c$. This can be separated to integrate,

$$\frac{1}{2\omega^2} = \alpha t + \frac{1}{2\omega_0} \tag{22}$$

where ω_0 is the value of the spin at time 0. The solution is

$$\omega(t) = \frac{\omega_0}{\sqrt{1 + 2\alpha t \omega_0^2}} \, . \tag{23}$$

(b) Now compute the radiation-reaction force on each particle. Use this to compute the torque on the system and so find another expression for $d\omega/dt$. Do they give the same result? Why or why not?

Solution: The Abraham-Dirac-Lorentz formula is

$$\boldsymbol{F}_{RR} = \frac{\mu_0 q^2}{6\pi c} \dot{\boldsymbol{a}} \, . \tag{24}$$

We have $\boldsymbol{a}$ pointing radially inward with magnitude $\omega^2 b$, so $\dot{\boldsymbol{a}}$ points in the opposite direction of velocity, with magnitude $\omega^3 b$. Each charged particle contributes to the torque the same amount, giving

$$\tau_{RR} = -2\frac{\mu_0 q^2}{6\pi c}\omega^3 b^2 \,. \tag{25}$$

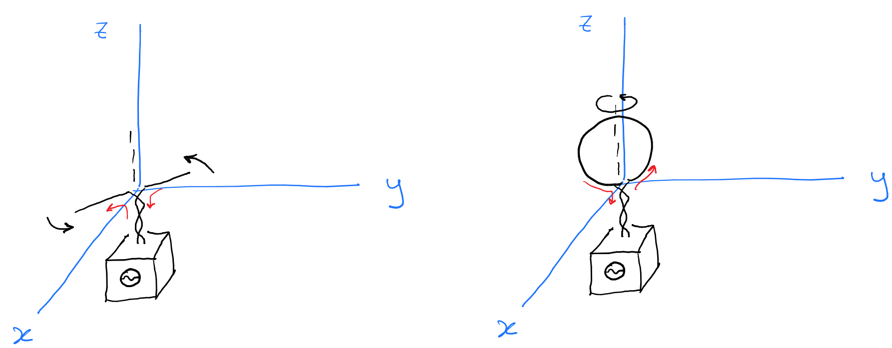
Torque is the time derivative of angular momentum, so we have

$$-\frac{\mu_0 q^2}{3\pi c}\omega^3 b^2 = \frac{d}{dt} I\omega = (2mb^2)\frac{d\omega}{dt} \,, \tag{26}$$

$$\implies \quad \frac{d\omega}{dt} = -\frac{\mu_0 q^2}{6\pi m c}\omega^3 \,. \tag{27}$$

Notice that this is off by a factor of 2 from the result we got in part (a). The calculation in this part is *only* from the radiation-reaction force, and did not include the Lorentz force on particle 1 due to the field of particle 2, and vice versa. That additional force accounts for the factor of 2. See the related discussion at the end of Griffiths' section 11.2.3.

4. **Two rotating antennas**. You might have seen a rotating antenna at an airport. Let's try to make two simple models. Suppose we take the simple split dipole antenna, at left, and the loop ("magnetic dipole") antenna, at right:



Each half of the split dipole has length b, and it is lying along direction $\hat{\boldsymbol{n}}$ somewhere in the $x-y$ plane. Meanwhile the loop antenna has radius b, and its normal vector $\hat{\boldsymbol{n}}$ lies somewhere in the $x-y$ plane.

In each scenario, the antenna is hooked up to a source of alternating current $I(t) = I_0 \cos(\Omega_s t)$, denoted with red arrows for one part of the oscillation. Each antenna is *also* hooked up to a motor that makes it rotate around the $\hat{\boldsymbol{z}}$ axis with a different angular frequency ω_r, denoted with the black arrows.

We're interested in describing the radiation at some point $\boldsymbol{r}$ at a large distance away from an antenna.

(a) In order to control the multipole expansion, we need to satisfy certain relationships between things like b, r, ω_r, and Ω_s. Explain which quantities must be very small or large, and why.

Solution: Based on the setup of the rotating antennas, we should naively expect radiation to come out at some combination of the frequencies ω_r and Ω_s. A closer examination would yield that the radiation will be at the two frequencies $|\omega_r \pm \Omega_s|$, but that level of detail is not need for this part. So, we need the following approximations:

i. $b \ll r$. The multipole expansion uses the dimensionless ratio b/r as an expansion parameter, so we need $b/r \ll 1$ to expand. In words, the field point $\boldsymbol{r}$ must be (far) outside of the source region.

ii. $b \ll c/\omega_r$. The source region must be small compared to the wavelength of radiation associated to frequency ω_r. Or, in other words, the rotation of the antenna geometry must happen on timescales much longer than $T_{l.c.} \equiv b/c$, the light-crossing time of the region, so we can always think of there being a well-defined "instantaneous" source geometry. This inequality also keeps the tips of the antennas moving much slower than the speed of light.

iii. $b \ll c/\Omega_s$. Just like above, the charges and currents must change slowly compared to $T_{l.c.}$, the light-crossing time of the region.

iv. $r \gg c/\omega_r$ and $r \gg c/\Omega_s$. The field point must be in the "wave zone" or "radiation zone," which is at least a few wavelengths of radiation away from the source. There can be radiation at either frequency ω_r or Ω_s (in more detail we see the true frequencies are $|\omega_r \pm \Omega_s|$).

For the remainder of the problem, you can assume that $\omega_r \ll \Omega_s$, if you didn't already assume that above.

(b) For the split dipole, what is the electric dipole $\boldsymbol{p}(t)$ as a function of time? For the loop antenna, what is the magnetic dipole $\boldsymbol{m}(t)$ as a function of time?

Solution: Given the current $I(t) = I_0 \cos(\Omega_s t)$, the charge that accumulates on one half of the split dipole is $q(t) = \frac{I_0}{\Omega_s} \sin(\Omega_s t)$. We can consider this charge located at location $\boldsymbol{\xi}_+(t) = \mathrm{Re}[be^{i\omega_r t}(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})]$, like in problem set 8 (there the rotation was clockwise, here it is anticlockwise as seen from above). Having the opposite charge at location $\boldsymbol{\xi}_- = -\boldsymbol{\xi}_+$ gives us the electric dipole moment

$$\boldsymbol{p}(t) = 2bq(t)\mathrm{Re}[e^{i\omega_r t}(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})] \tag{28}$$

$$\boldsymbol{p}(t) = \frac{2bI_0}{\Omega_s} \sin(\Omega_s t)\mathrm{Re}[e^{i\omega_r t}(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})]\,. \tag{29}$$

Here you can see the product of a sinusoid at frequency Ω_s and another at frequency ω_r, hence generating the two frequencies $|\omega_r \pm \Omega_s|$ from trig identities.

For the loop antenna, recall that for a planar circuit, the magnetic dipole moment is $\boldsymbol{m} = I\boldsymbol{a}$ where $\boldsymbol{a}$ points in the direction normal to the loop with magnitude equal to the area of the loop. Our current is oscillating and the normal is rotating, each at a different frequency. We have

$$\boldsymbol{m}(t) = I(t)\pi b^2 \mathrm{Re}[e^{i\omega_r t}(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})] \tag{30}$$

$$\boldsymbol{m}(t) = I_0 \cos(\Omega_s t)\pi b^2 \mathrm{Re}[e^{i\omega_r t}(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})]\,. \tag{31}$$

(c) For each of the two models (split dipole, loop antenna) give the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{B}$ at the very distant point $\boldsymbol{r}$. We only need the $\mathcal{O}(1/r)$ part, and you can use all the approximations above. [Hint: It will probably be simplest to first work out a coordinate-independent result, and then use it to find $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ components of the fields.]

Solution: For the electric dipole, we have the general coordinate-independent result (developed in Griffiths)

$$\boldsymbol{E}(\boldsymbol{r}, t) \simeq \frac{\mu_0}{4\pi r}[\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}})]\big|_{t=t_{ret}} \tag{32}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) \simeq -\frac{\mu_0}{4\pi rc}[\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}}]\big|_{t=t_{ret}}\,. \tag{33}$$

Here $\ddot{\boldsymbol{p}}$ needs to be evaluated at the retarded time, $t_0 = t - r/c$. So, let's compute this second time derivative, keeping in mind that we are assuming $\omega_r \ll \Omega_s$. Even though the time derivatives will act with the product rule, the Ω_s terms will be dominant over the ω_r terms, so we have

$$\ddot{\boldsymbol{p}}(t) \simeq -2bI_0\Omega_s \sin(\Omega_s t)\mathrm{Re}[e^{i\omega_r t}(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})] \tag{34}$$

$$\ddot{\boldsymbol{p}}(t) \simeq -2bI_0\Omega_s \sin(\Omega_s t)\mathrm{Re}[e^{i\omega_r t - i\phi}(\hat{\boldsymbol{r}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta - i\hat{\boldsymbol{\phi}})] \tag{35}$$

$$\ddot{\boldsymbol{p}}(t) \simeq -2bI_0\Omega_s \sin(\Omega_s t)[\cos(\omega_r t)\hat{\boldsymbol{x}} + \sin(\omega_r t)\hat{\boldsymbol{y}}]\,. \tag{36}$$

Now we need to take cross products, which can either be done using the $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ basis, or the $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ basis. The latter turns out to be simpler, using $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\theta}}$,

$$[\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}}] = -2bI_0\Omega_s \sin(\Omega_s t)\mathrm{Re}[e^{i\omega_r t - i\phi}(\hat{\boldsymbol{\phi}}\cos\theta + i\hat{\boldsymbol{\theta}})]\,, \tag{37}$$

$$\hat{\boldsymbol{r}} \times [\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}}] = -2bI_0\Omega_s \sin(\Omega_s t)\mathrm{Re}[e^{i\omega_r t - i\phi}(-\hat{\boldsymbol{\theta}}\cos\theta + i\hat{\boldsymbol{\phi}})]\,. \tag{38}$$

These expressions are then inserted into Eq. (32) and (33), keeping in mind that we need to evaluate $\ddot{\boldsymbol{p}}$ at time $t_0 = t - r/c$. Doing so we get

$$\boldsymbol{E}(\boldsymbol{r}, t) \simeq \frac{-2\mu_0 bI_0\Omega_s \sin(\Omega_s t_0)}{4\pi r}\mathrm{Re}[e^{i\omega_r t_0 - i\phi}(-\hat{\boldsymbol{\theta}}\cos\theta + i\hat{\boldsymbol{\phi}})]\,, \tag{39}$$

$$\simeq \frac{2\mu_0 bI_0\Omega_s \sin(\Omega_s t_0)}{4\pi r}[\cos(\omega_r t_0 - \phi)\cos\theta\hat{\boldsymbol{\theta}} + \sin(\omega_r t_0 - \phi)\hat{\boldsymbol{\phi}})]\,, \tag{40}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) \simeq \frac{2\mu_0 bI_0\Omega_s \sin(\Omega_s t_0)}{4\pi rc}\mathrm{Re}[e^{i\omega_r t_0 - i\phi}(\hat{\boldsymbol{\phi}}\cos\theta + i\hat{\boldsymbol{\theta}})]\,, \tag{41}$$

$$\simeq \frac{2\mu_0 bI_0\Omega_s \sin(\Omega_s t_0)}{4\pi rc}[\cos(\omega_r t_0 - \phi)\cos\theta\hat{\boldsymbol{\phi}} - \sin(\omega_r t_0 - \phi)\hat{\boldsymbol{\theta}})]\,. \tag{42}$$

Now we turn to the loop antenna. This is truly novel, since Griffiths did not develop the multipole expansion for an arbitrary source with no electric dipole radiation. So, let us go back to the Green's functions solutions for V and $\boldsymbol{A}$ in Lorenz gauge. Since $\rho = 0$ for the loop antenna (no charge accumulation anywhere), $V = 0$. For $\boldsymbol{A}$ in index notation we have

$$A^i = \frac{\mu_0}{4\pi}\int \frac{J^i(t_r, \boldsymbol{r}')}{\mathscr{r}}d^3\boldsymbol{r}'\,. \tag{43}$$

This is where we apply the multipole expansion, using the general result (proven on problem set 8)

$$\int \frac{S(t_r, \boldsymbol{r}')}{\mathscr{r}}d^3\boldsymbol{r}' = \frac{1}{r}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^n \int S(t_0, \boldsymbol{r}')(\hat{\boldsymbol{r}}\cdot\boldsymbol{r}')^n\, d^3\boldsymbol{r}' + \mathcal{O}(r^{-2})\,. \tag{44}$$

Let's examine the first two terms in A^i,

$$A^i = \frac{\mu_0}{4\pi r}\left[\int J^i(t_0, \boldsymbol{r}')d^3\boldsymbol{r}' + \frac{\partial}{\partial ct}\int J^i(t_0, \boldsymbol{r}')(\hat{\boldsymbol{r}}\cdot\boldsymbol{r}')d^3\boldsymbol{r}' + \dots\right]\,. \tag{45}$$

As discussed in class (and also an exercise in Griffiths), the first integral is $\dot{\boldsymbol{p}}(t_0)$, from a conservation law. This vanishes for the loop antenna because there is no electric dipole moment.
For the second integral, with the current restricted to a curve, so that we replace $\boldsymbol{J}d^3\boldsymbol{r}' \to I d\boldsymbol{l}'$, we need to compute

$$\oint (\hat{\boldsymbol{r}}\cdot\boldsymbol{r}')d\boldsymbol{l}'\,. \tag{46}$$

This is actually an exercise in Griffiths, and shown in his Eq. (5.82) [in the 3rd Edition]

$$\oint (\hat{\boldsymbol{r}}\cdot\boldsymbol{r}')d\boldsymbol{l}' = -\hat{\boldsymbol{r}} \times \int d\boldsymbol{a}'\,. \tag{47}$$

So, we can write this term in $\boldsymbol{A}$ using $\boldsymbol{m}$,

$$\boldsymbol{A} \simeq -\frac{\mu_0}{4\pi rc}\hat{\boldsymbol{r}} \times \dot{\boldsymbol{m}}\big|_{t=t_{ret}}\,. \tag{48}$$

Now computing $\boldsymbol{E} = -\nabla V - \partial\boldsymbol{A}/\partial t$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, we get

$$\boldsymbol{E} \simeq \frac{\mu_0}{4\pi rc}\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{m}}\big|_{t=t_{ret}}\,, \tag{49}$$

$$\boldsymbol{B} \simeq \frac{\mu_0}{4\pi rc^2}\hat{\boldsymbol{r}} \times [\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{m}}]\big|_{t=t_{ret}}\,. \tag{50}$$

You can check that the magnetic dipole radiation example worked out in the book is a special case of this general formula.

Now we plug in the magnetic dipole moment from before. These steps are very similar to the electric dipole part, again computing derivatives where Ω_s dominates over ω_r. We get

$$\ddot{\boldsymbol{m}}(t) = -I_0\Omega_s^2 \cos(\Omega_s t)\pi b^2 \mathrm{Re}[e^{i\omega_r t}(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})]\,, \tag{51}$$

$$\ddot{\boldsymbol{m}}(t) = -I_0\Omega_s^2 \cos(\Omega_s t)\pi b^2 \mathrm{Re}[e^{i\omega_r t - i\phi}(\hat{\boldsymbol{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta - i\hat{\boldsymbol{\phi}})]\,, \tag{52}$$

$$[\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{m}}] = -I_0\Omega_s^2 \cos(\Omega_s t)\pi b^2 \mathrm{Re}[e^{i\omega_r t - i\phi}(\hat{\boldsymbol{\phi}}\cos\theta + i\hat{\boldsymbol{\theta}})]\,, \tag{53}$$

$$\hat{\boldsymbol{r}} \times [\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{m}}] = -I_0\Omega_s^2 \cos(\Omega_s t)\pi b^2 \mathrm{Re}[e^{i\omega_r t - i\phi}(-\hat{\boldsymbol{\theta}}\cos\theta + i\hat{\boldsymbol{\phi}})]\,. \tag{54}$$

Finally, plugging into $\boldsymbol{E}$ and $\boldsymbol{B}$, making sure to evaluate at $t_0 = t - r/c$, find

$$\boldsymbol{E} \simeq \frac{-\mu_0 I_0 \Omega_s^2 \cos(\Omega_s t_0) b^2}{4rc} \mathrm{Re}[e^{i\omega_r t_0 - i\phi}(\hat{\boldsymbol{\phi}}\cos\theta + i\hat{\boldsymbol{\theta}})]\,, \tag{55}$$

$$\simeq \frac{-\mu_0 I_0 \Omega_s^2 \cos(\Omega_s t_0) b^2}{4rc} [\cos(\omega_r t_0 - \phi)\cos\theta\hat{\boldsymbol{\phi}} - \sin(\omega_r t_0 - \phi)\hat{\boldsymbol{\theta}}]\,, \tag{56}$$

$$\boldsymbol{B} \simeq \frac{-\mu_0 I_0 \Omega_s^2 \cos(\Omega_s t_0) b^2}{4rc^2} \mathrm{Re}[e^{i\omega_r t_0 - i\phi}(-\hat{\boldsymbol{\theta}}\cos\theta + i\hat{\boldsymbol{\phi}})]\,, \tag{57}$$

$$\simeq \frac{\mu_0 I_0 \Omega_s^2 \cos(\Omega_s t_0) b^2}{4rc^2} [\cos(\omega_r t_0 - \phi)\cos\theta\hat{\boldsymbol{\theta}} + \sin(\omega_r t_0 - \phi)\hat{\boldsymbol{\phi}}]\,. \tag{58}$$

5. Griffiths problem 12.47 (transform a plane electromagnetic wave to a new frame).

Solution:

(a) The electric and magnetic fields are

$$\boldsymbol{E} = E_0 \cos(kx - \omega t)\hat{\boldsymbol{y}}\,, \tag{59}$$

$$\boldsymbol{B} = \frac{E_0}{c} \cos(kx - \omega t)\hat{\boldsymbol{z}}\,, \tag{60}$$

where $k = \omega/c$.

(b) We use the general transformation of $\boldsymbol{E}, \boldsymbol{B}$ fields when one boosts in the x direction; but the only non-vanishing components we have to start with are E_y and B_z. The result is

$$\bar{E}_x = 0\,, \qquad \bar{E}_y = \gamma(E_y - vB_z)\,, \qquad \bar{E}_z = 0\,, \tag{61}$$

$$\bar{B}_x = 0\,, \qquad \bar{B}_y = 0\,, \qquad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y)\,. \tag{62}$$

Plugging in the fields from Eq. (59) and (60),

$$\bar{E}_y = \alpha E_0 \cos(kx - \omega t)\,, \qquad \bar{B}_z = \alpha \frac{E_0}{c} \cos(kx - \omega t)\,, \tag{63}$$

where

$$\alpha \equiv \gamma\left(1 - \frac{v}{c}\right) = \sqrt{\frac{1 - v/c}{1 + v/c}}\,. \tag{64}$$

Now we need to go to the barred spacetime coordinates via the Lorentz transformation, $x = \gamma(\bar{x} + v\bar{t})$, $t = \gamma(\bar{t} + \frac{v}{c^2}\bar{x})$, so

$$kx - \omega t = \bar{k}\bar{x} - \bar{\omega}\bar{t} \tag{65}$$

where

$$\bar{k} \equiv \gamma(k - \frac{\omega v}{c^2}) = \alpha k\,, \qquad \bar{\omega} \equiv \gamma\omega(1 - v/c) = \alpha\omega\,. \tag{66}$$

So in summary, we have

$$\bar{\boldsymbol{E}} = \bar{E}_0 \cos(\bar{k}\bar{x} - \bar{\omega}\bar{t})\hat{\boldsymbol{y}}\,, \tag{67}$$

$$\bar{\boldsymbol{B}} = \frac{\bar{E}_0}{c} \cos(\bar{k}\bar{x} - \bar{\omega}\bar{t})\hat{\boldsymbol{z}}\,, \tag{68}$$

where the barred quantities are $\bar{E}_0 = \alpha E_0$, $\bar{k} = \alpha k$, $\bar{\omega} = \alpha\omega$, where $\alpha = \sqrt{\frac{1-v/c}{1+v/c}}$.

(c) The new frequency is $\bar{\omega} = \alpha\omega = \sqrt{\frac{1-v/c}{1+v/c}}\omega$. This is the *Doppler shift*. The wavelength is $\bar{\lambda} = 2\pi/\bar{k} = 2\pi/\alpha k = \lambda/\alpha$. The phase velocity in this frame is $\bar{v} = \bar{\omega}/\bar{k} = c$. As required, the speed of light is the same in all inertial frames – that was the assumption for deriving the Lorentz transformations.

(d) Intensity is proportional to E^2, so the ratio of intensities is

$$\frac{\bar{I}}{I} = \frac{\bar{E}_0^2}{E_0^2} = \alpha^2 = \frac{1 - v/c}{1 + v/c}\,. \tag{69}$$

As you approach the speed of light, the amplitude, frequency, and intensity of light all go to zero, so it will be more and more difficult to see the light.