UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2020

Problem Set 9 — SOLUTIONS

Due: Sunday, May 3, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. Griffiths 11.15 (Peak emission angle and ultrarelativistic limit)

Solution: The maximum of the angular distribution occurs where the θ derivative vanishes,

$$\frac{d}{d\theta}\left[\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right]=0\,. \tag{1}$$

Taking the derivatives is straightforward,

$$0=\frac{2\sin\theta\cos\theta}{(1-\beta\cos\theta)^5}-\frac{5\beta\sin^3\theta}{(1-\beta\cos\theta)^6}\quad\Longrightarrow\quad 2\cos\theta(1-\beta\cos\theta)=5\beta\sin^2\theta=5\beta(1-\cos^2\theta)\,. \tag{2}$$

This is a quadratic equation for $\cos\theta$, which has solutions

$$\cos\theta_{\text{max}}=\frac{-2\pm\sqrt{4+60\beta^2}}{6\beta}=\frac{-1\pm\sqrt{1+15\beta^2}}{3\beta}\,. \tag{3}$$

We can see that the negative root is unphysical for all values of β because it is always less than -1. So, the peak of the radiation is at angle

$$\cos\theta_{\text{max}}=\frac{-1+\sqrt{1+15\beta^2}}{3\beta}\,. \tag{4}$$

Now to get approximations in the $\beta\to 1$ limit, let's define a small quantity $\epsilon\equiv 1-\beta$, $\epsilon\ll 1$, and do Taylor expansions in ϵ. The expansions we need are $1/(1+x)\approx 1-x$, $\sqrt{1+x}\approx 1+\frac{x}{2}$, and $\cos x=1-\frac{x^2}{2}$, where $x\ll 1$ in all of these expansions. Using this for Eq. (4), we get

$$\cos\theta_{\text{max}}\approx 1-\frac{1}{2}\theta^2_{\text{max}}\approx\frac{-1+\sqrt{1+15(1-\epsilon)^2}}{3(1-\epsilon)}\approx\frac{1}{3}(1+\epsilon)(-1+\sqrt{16-30\epsilon}) \tag{5}$$

$$1-\frac{1}{2}\theta^2_{\text{max}}\approx\frac{1}{3}(1+\epsilon)\left[-1+4\left(1-\frac{30}{32}\epsilon\right)\right]\approx 1-\frac{1}{4}\epsilon\,. \tag{6}$$

Finally we find

$$\theta_{\text{max}}\approx\sqrt{\frac{\epsilon}{2}}=\sqrt{\frac{1-\beta}{2}}\,. \tag{7}$$

Finally we want to know what is the ratio of (peak $dP/d\Omega$ at ultrarelativistic speeds) over (peak $dP/d\Omega$ at rest), for fixed charge and acceleration. At rest ($\beta=0$), $dP/d\Omega|_{\theta_{\text{max}}}=\mu_0q^2a^2/16\pi^2c$, because the peak is at $\theta_{\text{max}}=\pi/2$. This cancels the prefactor so we just need to evaluate

$$\frac{\sin^2\theta_{\text{max}}}{(1-\beta\cos\theta_{\text{max}})^5}\approx\frac{\epsilon/2}{(1-(1-\epsilon)(1-\epsilon/4))^5}\approx\left(\frac{4}{5}\right)^5\frac{1}{2\epsilon^4}\,. \tag{8}$$

If we want to express this in terms of γ, we note that in the ultrarelativistic limit,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx \frac{1}{\sqrt{2\epsilon}} \quad \Longrightarrow \quad \epsilon \approx \frac{1}{2\gamma^2}\,. \tag{9}$$

So in terms of γ, the ratio of peak intensities is

$$\frac{\sin^2\theta_{\rm max}}{(1-\beta\cos\theta_{\rm max})^5} \approx \frac{1}{4}\left(\frac{8}{5}\right)^5 \gamma^8 \approx 2.62\gamma^8\,. \tag{10}$$

2. Griffiths 11.31 (Radiation and reaction for a particle in hyperbolic motion)

Solution: The particle in hyperbolic motion follows the trajectory

$$w = \sqrt{b^2+c^2t^2}\,, \tag{11}$$

with velocity

$$v = \frac{dw}{dt} = \frac{c^2 t}{\sqrt{b^2+c^2t^2}}\,. \tag{12}$$

The Lorentz factor squared is

$$\gamma^2 = \frac{1}{1-v^2/c^2} = \frac{b^2+c^2t^2}{b^2}\,. \tag{13}$$

The acceleration is

$$a = \frac{dv}{dt} = \frac{c^2}{\sqrt{b^2+c^2t^2}} - \frac{c^4t^2}{(b^2+c^2t^2)^{3/2}} = \frac{b^2c^2}{(b^2+c^2t^2)^{3/2}}\,. \tag{14}$$

We will also need the "jerk"

$$\dot{a} = \frac{-3b^2c^4t}{(b^2+c^2t^2)^{5/2}}\,. \tag{15}$$

These are all the ingredients we need to see if it radiates, and if there is radiation-reaction force.

(a) The radiation power is

$$P = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c} = \frac{\mu_0 q^2}{6\pi c}\frac{b^4c^4}{(b^2+c^2t^2)^3}\frac{(b^2+c^2t^2)^3}{b^6} = \frac{q^2 c}{6\pi\epsilon_0 b^2} \tag{16}$$

which is not zero, so yes, the particle radiates.

(b) The radiation-reaction force is given by

$$F_{RR} = \frac{\mu_0 q^2\gamma^4}{6\pi c}\left(\dot{a} + \frac{3\gamma^2a^2v}{c^2}\right)\,. \tag{17}$$

We know the prefactor does not vanish, but the term in parentheses is

$$\dot{a} + \frac{3\gamma^2a^2v}{c^2} = \frac{-3b^2c^4t}{(b^2+c^2t^2)^{5/2}} + 3\frac{b^2+c^2t^2}{b^2}\left(\frac{b^2c^2}{(b^2+c^2t^2)^{3/2}}\right)^2\frac{t}{\sqrt{b^2+c^2t^2}} = 0\,. \tag{18}$$

Therefore there is *no radiation-reaction force.* To fully understand the implication here requires learning general relativity!

3. Griffiths 12.5 ("Seeing" vs. "observing" a row of synchronized clocks)

Solution:

(a) The distance to the 90th clock is $d = 90 \times 10^6$km, so the light travel time is $t = d/c = 5$min. Therefore the light you *see* at noon left at 11:55 am.

(b) Since you know the distance, you take this into account in your observation, so you "observe" noon.

4. Griffiths 12.7 (Observed lifetime of a muon)

Solution: Time dilation has not been taken into account. The lifetime $\tau \approx 2 \times 10^{-6}$s is in the rest frame of the muon. In the lab frame, this will be observed as $t = \gamma\tau$ where $\gamma = 1/\sqrt{1 - v^2/c^2}$ comes from the muon's velocity v. Instead of the student's wrong equation for velocity ($v \neq d/\tau$ with $d = 800$m), we should have

$$v = \frac{d}{\gamma\tau} = \frac{d}{\tau}\sqrt{1 - v^2/c^2}\,. \tag{19}$$

Now you can solve for v, finding

$$v = \frac{1}{\sqrt{(\tau/d)^2 + 1/c^2}} = \frac{4}{5}c\,. \tag{20}$$

5. Griffiths 12.26: Find the invariant product of the 4-velocity with itself, $\eta^\mu\eta_\mu$.

Solution:

$$\begin{aligned}\eta^\mu\eta_\mu &= \eta^0\eta_0 + \eta^1\eta_1 + \eta^2\eta_2 + \eta^3\eta_3 \\ &= -(\eta^0)^2 + (\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2 \\ &= \frac{1}{1 - u^2/c^2}\left[-c^2 + u^2\right] = -c^2\,.\end{aligned}$$

In a sense, this says that everyone is moving through spacetime at the same speed (1 second per second as measured on their own clock).