UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2020

Problem Set 8

Due: Saturday, Apr. 25, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. **Radiation from charges in circular motion**. Suppose we have a rotor of length $2b$ that lies in the $x-y$ plane, centered on the $\hat{z}$ axis. We attach a charge $+q$ to one end, and an opposite charge $-q$ to the opposite end, and then spin this rotor around the $\hat{z}$ axis (clockwise as seen from above) at an angular frequency of ω radians per second. At time $t=0$ it lies along the x axis.

   (a) Write the trajectories $\boldsymbol{\xi}_+(t)$ of the positive charge and $\boldsymbol{\xi}_-(t)$ of the negative charge. What is the charge distribution $\rho(t, \boldsymbol{r})$? (There will be delta functions!)

   (b) Compute the first three charge *moments* of this distribution, as a function of time: the 0th moment (charge), the 1st moment (charge dipole), and the 2nd moment (charge quadrupole). The charge quadrupole moment $\overleftrightarrow{\boldsymbol{\mathcal{Q}}}$ is a symmetric tensor defined similarly to the charge monopole Q and dipole moment $\boldsymbol{p}$,

$$Q(t) = \int \rho(t, \boldsymbol{r}')d^3\boldsymbol{r}' \tag{1}$$

$$p_i(t) = \int \rho(t, \boldsymbol{r}')r'_i d^3\boldsymbol{r}' \tag{2}$$

$$\mathcal{Q}_{ij}(t) = \int \rho(t, \boldsymbol{r}')r'_i r'_j d^3\boldsymbol{r}' \,. \tag{3}$$

   [**Note** 1: The definition of quadrupole moment I wrote here differs from what Griffiths writes in Chapter 3. The convention I'm using is more in line with what's in the general relativity literature. Note 2: If you find this problem confusing, you probably want to review chapter 3, and perhaps try problem 3.45.]

   (c) How would you define the octupole moment of a charge distribution?

   (d) Find the second time derivative of the dipole moment, $\ddot{\boldsymbol{p}}$.

   (e) Using Griffiths' result for the dipole radiation from a general charge distribution, find the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{B}$ produced by this system, at a large distance $r \gg \lambda$. [Note that Griffiths' equation in terms of $\ddot{\boldsymbol{p}}$ is correct for any orientation of coordinate system. However his later equations in terms of $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ have aligned the z axis along the (retarded) direction of $\ddot{\boldsymbol{p}}$, so if you wanted to apply those, you'd need to constantly change your basis!]

   (f) What is the polarization of the electromagnetic radiation along the positive z axis? Negative z axis? What about in the $x-y$ plane?

   (g) Find the Poynting vector $\boldsymbol{S}$, and its period average $\langle \boldsymbol{S} \rangle$ at this large distance. Compare the angular dependence of the radiation pattern in this system with the one we saw before (a center-fed split dipole antenna).

   (h) What is the total power emitted in radiation?

   (i) What would happen if you replaced the negative charge with another positive charge?

2. **Getting higher multipoles**. In this problem we will extend the derivation from dipole to higher multipole order. We will do so for the scalar wave equation,

$$\Box f(t, \boldsymbol{r}) = S(t, \boldsymbol{r})\,, \tag{4}$$

which has the solution

$$f(t, \boldsymbol{r}) = \frac{-1}{4\pi} \int \frac{S(t_r, \boldsymbol{r}')}{\mathscr{r}} d^3\boldsymbol{r}'\,, \tag{5}$$

where as before $t_r = t - \frac{1}{c}\mathscr{r}$.

   (a) Show how to perform a Taylor series expansion on the first argument of S, so that instead of evaluating S at different retarded times, we evaluate various time derivatives of S all at the single time $t_0 \equiv t - \frac{r}{c}$.

   (b) In the above Taylor series, you should see the combination $(r - \mathscr{r})$ appearing. Using the series expansion $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(x^3)$, find the first two nonvanishing terms of $(r - \mathscr{r})$. This will be in terms of r, r', and $\cos\theta' = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}'$.

   (c) Combine your results to prove that the $\frac{1}{r}$ part of the solution for f is given by

$$f(t, \boldsymbol{r}) = \frac{-1}{4\pi}\frac{1}{r}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^n \int S(t_0, \boldsymbol{r}')(\hat{\boldsymbol{r}} \cdot \boldsymbol{r}')^n \, d^3\boldsymbol{r}' + \mathcal{O}(r^{-2})\,. \tag{6}$$

   The nth term in the sum is determined by n derivatives of the nth multipole moment of the source.