UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2020

Problem Set 6 — SOLUTIONS

Due: Wednesday, Apr. 1, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. **Energy/momentum transfer**. Suppose a plane electromagnetic wave is traveling through vacuum (constants ϵ_0 and μ_0) in the $\hat{z}$ direction, with frequency ω, with the electric field linearly polarized in the $\hat{x}$ direction with amplitude E_{0I}.

   (a) This wave is incident on a perfectly absorbing sheet lying in the $x-y$ plane. How much energy does the sheet absorb per unit time, per unit area? How much momentum?

   Solution: The sheet absorbs all the energy carried by the wave, computed from the energy flux (Poynting vector),

$$\frac{dE}{dtdA} = \langle \boldsymbol{S} \cdot \hat{\boldsymbol{n}} \rangle = cu = \frac{1}{2} c\epsilon_0 E_{0I}^2 \,, \tag{1}$$

   where $\hat{\boldsymbol{n}}$ is the normal to the surface, which we take to be collinear with the direction of propagation. The momentum transfer actually comes from $-\langle \overleftrightarrow{\boldsymbol{T}} \cdot \hat{\boldsymbol{n}} \rangle$, but you can also follow Griffiths and get the same result. The result is

$$\frac{d\boldsymbol{p}}{dtdA} = -\langle \overleftrightarrow{\boldsymbol{T}} \cdot \hat{\boldsymbol{n}} \rangle = \hat{\boldsymbol{z}} \frac{I}{c} = \frac{1}{2} \epsilon_0 E_{0I}^2 \hat{\boldsymbol{z}} \,, \tag{2}$$

   which must be written as a vector!

   (b) Now suppose we replace the perfect absorber with a perfectly reflecting mirror. How much energy does the mirror absorb per unit time, per unit area? How much momentum?

   Solution: If incident on a perfect reflector, all of the incident energy becomes reflected energy, so since energy is conserved, there is no energy absorbed. The momentum flux in the incident wave is $d\boldsymbol{p}_I/dtdA = \frac{1}{2}\epsilon_0 E_{0I}^2 \hat{\boldsymbol{z}}$, and if we perfectly reflect it then the momentum flux in the reflected wave will have the opposite sign, $d\boldsymbol{p}_R/dtdA = -\frac{1}{2}\epsilon_0 E_{0I}^2 \hat{\boldsymbol{z}}$. But momentum must also be conserved, so the mirror must have absorbed enough momentum so that the total momentum is the same as the initial. Therefore the mirror absorbed the difference,

$$\frac{d\boldsymbol{p}_M}{dtdA} = \epsilon_0 E_{0I}^2 \hat{\boldsymbol{z}} \,. \tag{3}$$

   Next suppose we replace the perfect reflector with a partially-transmitting sheet of linear medium with electric permittivity ϵ_2 and magnetic permeability μ_0. This sheet has some finite thickness but let us focus only on the first interface, between vacuum and the material, and ignore everything that happens downstream.

   (c) For normal incidence, what are the reflected and transmitted electric and magnetic fields in terms of the incident field?

   Solution: This was worked out in Griffiths. Below are the complex fields; the physical fields are the real part.

$$\tilde{\boldsymbol{E}}_R = \tilde{E}_{0R} e^{i(-kz-\omega t)} \hat{\boldsymbol{x}} \tag{4}$$
$$\tilde{\boldsymbol{B}}_R = -\frac{1}{c} \tilde{E}_{0R} e^{i(-kz-\omega t)} \hat{\boldsymbol{y}} \tag{5}$$
$$\tilde{\boldsymbol{E}}_T = \tilde{E}_{0T} e^{i(k_2 z-\omega t)} \hat{\boldsymbol{x}} \tag{6}$$
$$\tilde{\boldsymbol{B}}_T = \frac{1}{v_2} \tilde{E}_{0T} e^{i(k_2 z-\omega t)} \hat{\boldsymbol{y}}\,, \tag{7}$$

where $k = \omega/c$, $v_2^2 = 1/\epsilon_2\mu_2$, $k_2 = \omega/v_2$, and where

$$\tilde{E}_{0R} = \frac{1-\beta}{1+\beta} \tilde{E}_{0I} \tag{8}$$
$$\tilde{E}_{0T} = \frac{2}{1+\beta} \tilde{E}_{0I}\,, \tag{9}$$

with $\beta \equiv \mu_1 v_1/\mu_2 v_2$.

(d) What is the momentum density $\wp$ (which is real, not complex) in the transmitted field? (Hint: how do the permittivity and permeability enter into $\wp$?)

Solution: First we get the Poynting flux in a linear medium, $\boldsymbol{S} = \frac{1}{\mu}\boldsymbol{E} \times \boldsymbol{B}$; and then if we redo the derivation of linear momentum density in this same medium, we get

$$\wp = \epsilon\mu \boldsymbol{S} = \epsilon \boldsymbol{E} \times \boldsymbol{B}\,. \tag{10}$$

Plugging in the above electric and magnetic fields we find

$$\langle \wp_T \rangle = \frac{\epsilon}{v_2} \frac{1}{2} E_{0T}^2 \hat{\boldsymbol{z}} = \frac{\epsilon}{v_2} E_{0I}^2 \frac{2}{(1+\beta)^2} \hat{\boldsymbol{z}}\,. \tag{11}$$

(e) What is the momentum density, separately, in (i) the incident field, and (ii) the reflected field?

Solution: These waves are in vacuum, so

$$\wp = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B}\,, \tag{12}$$
$$\langle \wp_I \rangle = \frac{1}{2c} \epsilon_0 E_{0I}^2 \hat{z}\,, \tag{13}$$
$$\langle \wp_R \rangle = -\frac{1}{2c} \epsilon_0 E_{0R}^2 \hat{z} = -\frac{1}{2c} \epsilon_0 E_{0I}^2 \frac{(1-\beta)^2}{(1+\beta)^2} \hat{\boldsymbol{z}}\,. \tag{14}$$

(f) What is the momentum density in the *sum* of the incident and reflected fields?

Solution: Now we try to do the average of the combined fields. The complex fields are

$$\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_I + \tilde{\boldsymbol{E}}_R = E_{0I} \left(e^{ikz} + \frac{1-\beta}{1+\beta} e^{-ikz} \right) e^{i\omega t} \hat{\boldsymbol{x}}\,, \tag{15}$$
$$\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}_I + \tilde{\boldsymbol{B}}_R = \frac{1}{c} E_{0I} \left(e^{ikz} - \frac{1-\beta}{1+\beta} e^{-ikz} \right) e^{i\omega t} \hat{\boldsymbol{y}}\,. \tag{16}$$

Take the real parts, multiply everything out in $\wp = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B}$ (since we are in vacuum), and average over a period. Then you will find

$$\langle \wp_{\text{tot}} \rangle = \frac{\epsilon_0}{c} E_{0I}^2 \frac{2\beta}{(1+\beta)^2} \hat{\boldsymbol{z}}\,. \tag{17}$$

In particular note that if you take the previous result, you will find

$$\langle \wp_I \rangle + \langle \wp_R \rangle = \langle \wp_{\text{tot}} \rangle\,, \tag{18}$$

which doesn't really have to be the case, since $\wp$ is a nonlinear function of the fields.

(g) Which of these results do you think is the correct way to compute the momentum transferred to the partially-transmitting sheet? Justify your claim.

Solution: We want this result to agree with the case of reflection off of a perfect mirror, where we had that $\boldsymbol{p}_I = \boldsymbol{p}_R + \boldsymbol{p}_{\text{mirror}}$. In the partially transmitting case, we should compute the mirror's momentum via $\boldsymbol{p}_I = \boldsymbol{p}_T \boldsymbol{p}_R + \boldsymbol{p}_{\text{mirror}}$. This is easiest to justify by thinking in the time domain instead of frequency domain (which is the steady-state situation). Imagine sending in a short pulse of electromagnetic radiation. At early times there will be an incident wave, having only $\boldsymbol{p}_I$. At late times there will be a separate reflected pulse and transmitted pulse, with a momentum mismatch – whose imbalance will tell us how much momentum must have been transferred to the mirror.

2. **Reflection with horizontal polarization** (long). In lecture we went through the derivation of reflection where the electric field is "vertically" polarized, i.e. with the $\boldsymbol{E}$ field lying in the $x - z$ plane of incidence. Redo the calculation but with the horizontal polarization, i.e. with $\boldsymbol{E} \propto \hat{y}$.

(a) Write down the four boundary conditions, evaluated with the appropriate parallel/perpendicular electric and magnetic fields, in terms of the angles θ_I, θ_T etc.

Solution: We will have the fields

$$\tilde{\boldsymbol{E}}_I = \tilde{E}_{0I} e^{i(\boldsymbol{k}_I \cdot \boldsymbol{r} - \omega t)} \hat{\boldsymbol{y}} \tag{19}$$

$$\tilde{\boldsymbol{B}}_I = \frac{1}{v_1} \tilde{E}_{0I} e^{i(\boldsymbol{k}_I \cdot \boldsymbol{r} - \omega t)} (-\cos\theta_I \hat{\boldsymbol{x}} + \sin\theta_I \hat{\boldsymbol{z}}) \tag{20}$$

$$\tilde{\boldsymbol{E}}_R = \tilde{E}_{0R} e^{i(\boldsymbol{k}_R \cdot \boldsymbol{r} - \omega t)} \hat{\boldsymbol{y}} \tag{21}$$

$$\tilde{\boldsymbol{B}}_R = \frac{1}{v_1} \tilde{E}_{0R} e^{i(\boldsymbol{k}_R \cdot \boldsymbol{r} - \omega t)} (\cos\theta_I \hat{\boldsymbol{x}} + \sin\theta_I \hat{\boldsymbol{z}}) \tag{22}$$

$$\tilde{\boldsymbol{E}}_T = \tilde{E}_{0T} e^{i(\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega t)} \hat{\boldsymbol{y}} \tag{23}$$

$$\tilde{\boldsymbol{B}}_T = \frac{1}{v_2} \tilde{E}_{0T} e^{i(\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega t)} (-\cos\theta_T \hat{\boldsymbol{x}} + \sin\theta_T \hat{\boldsymbol{z}}) \tag{24}$$

where we already know that $\theta_I = \theta_R$ and the law of refraction ($\sin\theta_T / \sin\theta_I = v_2/v_1$). We can impose the four boundary conditions

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \tag{25a}$$

$$\boldsymbol{E}_1^{\parallel} = \boldsymbol{E}_2^{\parallel} \tag{25b}$$

$$B_1^{\perp} = B_2^{\perp} \tag{25c}$$

$$\frac{1}{\mu_1} \boldsymbol{B}_1^{\parallel} = \frac{1}{\mu_2} \boldsymbol{B}_2^{\parallel} \tag{25d}$$

The first is automatically satisfied. The second tells us $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$. The third tells us the same thing, after using the law of refraction (or alternatively, this is where the law comes from). The fourth tells us

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \left(\frac{\mu_1 v_1 \cos\theta_T}{\mu_2 v_2 \cos\theta_I} \right) \tilde{E}_{0T} \,. \tag{26}$$

(b) Solve the resulting linear system for the ratios $\tilde{\boldsymbol{E}}_{0R}/\tilde{\boldsymbol{E}}_{0I}$ and $\tilde{\boldsymbol{E}}_{0T}/\tilde{\boldsymbol{E}}_{0I}$, in terms of the earlier variables $\alpha \equiv \cos\theta_T / \cos\theta_I$ and $\beta \equiv \mu_1 v_1 / \mu_2 v_2$.

Solution: The solution to the system is

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{1 - \alpha\beta}{1 + \alpha\beta} \tag{27}$$

$$\frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = \frac{2}{1 + \alpha\beta} \,. \tag{28}$$

(c) Give an equation for Brewster's angle θ_B where there is no (horizontal) reflection. Assuming that $\mu_1 \approx \mu_2$ to simplify, what do you find for the no-reflection condition?

Solution: No reflection occurs when $\alpha\beta = 1$. In terms of angles, this happens when

$$\alpha = \frac{\sqrt{1 - \sin^2\theta v_2^2/v_1^2}}{\cos\theta} = \frac{1}{\beta} = \frac{\mu_2 v_2}{\mu_1 v_1} \tag{29}$$

or

$$1 = \left(\frac{v_2}{v_1}\right)^2 [\sin^2\theta + \frac{\mu_2^1}{\mu_1^2}\cos^2\theta] \,. \tag{30}$$

If $\mu_1 \approx \mu_2$, this only happens when $v_1 \approx v_2$, but that just means the two media are optically identical, so there wouldn't be any reflection.

(d) Check that the reflection and transmission coefficients add up to 1 (recall that the transmission coefficient is the ratio of intensities, rather than the square of the ratio of electric fields).

Solution: The reflection and transmission coefficients are

$$R = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 , \qquad T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\alpha\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2 . \tag{31}$$

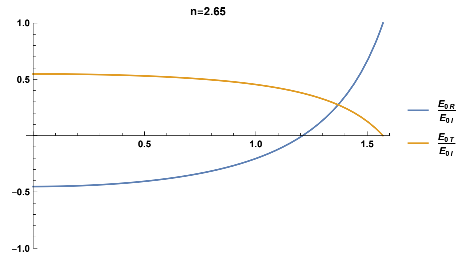
The sum is indeed 1,

$$R + T = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 + \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2 = \frac{1 - 2\alpha\beta + \alpha^2\beta^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = 1\,. \tag{32}$$

3. Silicon carbide has an index of refraction of $n = 2.65$.

(a) Plot the ratios E_{0R}/E_{0I} and E_{0T}/E_{0I} as a function of θ_I, for the interface between SiC and air (assuming $\mu_1 = \mu_2 = \mu_0$).

Solution:



(b) What are the values for the two amplitude ratios at normal incidence?

Solution: At $\theta = 0$, $E_{0R}/E_{0I} \approx -0.45$ and $E_{0T}/E_{0I} \approx +0.55$.

(c) What is Brewster's angle?

Solution: $\theta_B \approx 69.3° \approx 1.21$ rad.

(d) What is the "crossover" angle, where the reflection and transmission amplitudes are equal?

Solution: $\theta_x \approx 78.5° \approx 1.37$ rad.

4. Griffiths 9.22a-b (phase and group velocities in deep water waves, and quantum mechanics). Note that Griffiths writes "wave velocity" for what everyone calls the *phase* velocity.

Solution:

(a) Deep water waves satisfy

$$v_{ph} = \frac{\omega}{k} \propto \sqrt{\lambda}\,. \tag{33}$$

Therefore we have $\omega = Ck^{1/2}$ for some constant C. Then we can find the group velocity as

$$v_g = \frac{d\omega}{dk} = \frac{C}{2\sqrt{k}} = \frac{1}{2}\frac{\omega}{k} = \frac{1}{2}v_{ph}\,. \tag{34}$$

(b) In the phase factor, we identify $kx - \omega t = px - Et$. So we find $k = p$ and $\omega = E = p^2/2m = k^2/2m$. From this dispersion relationship we find the phase and group velocities,

$$v_{ph} = \frac{\omega}{k} = \frac{k}{2\pi} \tag{35}$$

$$v_g = \frac{d\omega}{dk} = \frac{k}{\pi} = 2v_{ph}. \tag{36}$$

The classical particle speed is $v = p/m = k/m = v_g$.