UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2020

Problem Set 5 — MIDTERM

Due: Wednesday, Mar. 18, 2020, by 5PM

Material: The midterm covers the material so far except for last week and this week (linear oscillators).

Due date: Wednesday, Mar. 18, 2020 by 5PM to 205 Lewis Hall. If my door is closed, please slide the exam under my door. Late exams will require extenuating circumstances.

Logistics: The exam consists of this page plus two pages of questions. Do not look at the problems until you are ready to start it.

Time: The work might expand to eat up as much time as you allot – therefore I highly recommend you restrict yourself to no more than 5 hours cumulative time spent on these problems. You may take as many breaks as you like, not counted against the 5 hours. **You should not be consulting references, working on the problems, or discussing with others during the breaks**.

Resources: The midterm and final are **not collaborative**. All questions must be done on your own, without consulting anyone else. You may consult your own notes (both in-class and notes on this class you or a colleague in the class have made), the textbook by Griffiths, and solution sets on the course website. **You may not consult any other material**, including other textbooks, the web (except for the current Phys. 402 website), material from previous years' Phys. 402 or any other classes, or copies you have made of such material, or any other resources. Calculators and symbolic manipulation programs are not allowed.

1. **Falling conducting ring**. Let's model the Earth as a perfect sphere, and model its magnetic field (above the ground) as a pure magnetic dipole given by moment $\boldsymbol{m}_E$, located at the center of the sphere.

   (a) Write down a coordinate-independent expression for this $\boldsymbol{B}$ field.

   Now suppose we have a conducting circular ring of radius s and conductivity σ. The ring is made of wire whose cross-sectional area is A.

   (b) In terms of the given quantities, what is the ring's total resistance R?

   Now let's say we're standing on the Earth at a colatitude θ measured from the magnetic north pole. We're holding this ring so that its symmetry axis is parallel to the ground, and is pointing toward magnetic north. We drop it from distance r from the center of the Earth.

   (c) Find the magnetic flux Φ as a function of r and θ.

   (d) Assume the motion of the ring is dominated by the force of gravity, and find the EMF through the ring.

   Suppose the ring has a self-inductance L. Now you can model the ring as a circuit with a source of EMF, a resistor, and an inductor.

   (e) Draw the circuit diagram.

   (f) Write down a differential equation for the current as a function of time. Solve for $I(t)$ (assuming the ring will only fall a short distance, not through the Earth!). From the point of view of an observer looking at the ring from a point closer to the magnetic north pole, is the current flowing clockwise or anti-clockwise?

   Now that there is a current flowing in the ring, we can model it as a small magnetic dipole.

   (g) What is its dipole moment $\boldsymbol{m}_{\text{ring}}$?

   (h) What is the torque (both magnitude and direction) on the ring? Describe which way it will rotate (a diagram may be helpful).

   (i) What is the magnetic force on the ring (magnitude and direction)?

2. **Magnetic field between two coils**. We have two circular current loops in our lab, each of radius R. We place them both on the z axis with their symmetry axes also along z. One of them is at height $z = +a$, and the other is at height $z = -a$. We run the same current I through both coils.

   (a) Write down an integral for the mutual inductance between the two loops. (This should be an ordinary one-dimensional integral; make as much progress as possible but you may not be able to completely evaluate it)

   (b) What is the magnetic field along the z axis, $\boldsymbol{B}(\rho = 0, z)$, in the region between the two loops? Show that $\left.\frac{\partial B_z}{\partial z}\right|_{z=0} = 0$.

   (c) Find the value of a that will make the z component of the magnetic field more uniform at the center, so as to satisfy

   $$\left.\frac{\partial^2 B_z}{\partial z^2}\right|_{z=0} = 0\,.$$

3. **Taking a few derivatives**. Suppose we create an electromagnetic field given by

   $$\boldsymbol{E} = (\beta \sin y\, \hat{\boldsymbol{x}} + \beta \cos x\, \hat{\boldsymbol{y}} + \alpha \sin z\, \hat{\boldsymbol{z}})e^{-t}\,,$$
   $$\boldsymbol{B} = -\beta(\cos y + \sin x)e^{-t}\, \hat{\boldsymbol{z}}\,.$$

   (a) Show that $\nabla \cdot \boldsymbol{B} = 0$ is satisfied.

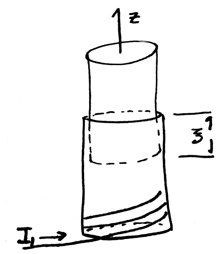
   (b) Show that $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ is satisfied.

(c) What charge density ρ and current density $\boldsymbol{J}$ would produce this electromagnetic field?

(d) Show that $\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$ is satisfied.

(e) Compute the Poynting vector $\boldsymbol{S}$ of this field configuration.

4. **Partially overlapping solenoids**. Suppose we have two almost-identical solenoids: they have length ℓ, with N total turns, giving $n \equiv N/\ell$ turns per unit length, and the wire in each solenoid has a total resistance R. One of them has radius $r + \epsilon$ and the other has radius r, so they can just barely nest within each other (but for calculation purposes we can consider both of them to have radius r).



We place both of these solenoids along the z axis, with a length ξ of overlap (see figure). However this overlap is not fixed – the two solenoids can move in response to forces, and thus $\xi(t)$ may depend on time.

Compute the mutual inductance $M(\xi)$ of the two solenoids, which will depend on ξ (you can assume that when you run current through one of the solenoids, it will create a uniform magnetic field inside and no magnetic field outside).

5. **Magnetic fluid in a solenoid**. Let's take a solenoid of radius r and height h, with $n \equiv N/h$ turns per unit length. We hold this solenoid vertically just on the surface of a bath of incompressible fluid of density ρ. This fluid is a linear magnetic medium with magnetic susceptibility χ_m, either positive or negative. Suppose that at some instant, this fluid also fills the solenoid up to height ζ (i.e. the inside of the solenoid has fluid in the range $0 \leq z \leq \zeta$, and it has air in the range $\zeta \leq z \leq h$).

(a) What is the magnetic field $\boldsymbol{B}$ in each of the two regions (fluid and air)?

(b) Find the total self-inductance L of the partially-filled solenoid, as a function of ζ.

(c) Find the energy stored in the magnetic field as a function of ζ. There are two (or more) ways to do this – check the correctness of your result by showing both calculations.

(d) If we turn on a current in the solenoid, the fluid may be pulled upward, depending on the sign of the susceptibility. Will a positive or negative χ_m be pulled upward?

(e) There is an energy cost to raising this fluid to height ζ. Write the total energy (gravitational and magnetic). Find the equilibrium height ζ_{eq} which the fluid achieves as a function of the current I (and any other quantities in the problem).

6. **Standing electromagnetic wave**. Suppose we superimpose the following two complex, monochromatic waves, both with frequency ω: one traveling in the $\hat{z}$ direction, and the electric field is polarized in $\hat{x}$ with amplitude $\tilde{E}_0$; and the second is traveling in the $-\hat{z}$ direction, with electric field polarized in $-\hat{x}$ with the same amplitude $\tilde{E}_0$.

(a) Write down the complex $\tilde{\boldsymbol{E}}, \tilde{\boldsymbol{B}}$ fields for each of the two waves separately.

(b) Now superimpose them and find the *real* field that arises from the superposition.

(c) Find the part of the period-averaged energy density due just to the electric field and comment on its pattern in space.

(d) Now examine the total period-averaged energy density $\langle u \rangle$ (due to both the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields), and again comment on its pattern in space.

(e) Finally, find the period-averaged Poynting vector, $\langle \boldsymbol{S} \rangle$.