UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2020

Problem Set 5 — SOLUTIONS

Due: Wednesday, Mar. 18, 2020, by 5PM

Material: The midterm covers the material so far except for last week and this week (linear oscillators).

Due date: Wednesday, Mar. 18, 2020 by 5PM to 205 Lewis Hall. If my door is closed, please slide the exam under my door. Late exams will require extenuating circumstances.

Logistics: The exam consists of this page plus two pages of questions. Do not look at the problems until you are ready to start it.

Time: The work might expand to eat up as much time as you allot – therefore I highly recommend you restrict yourself to no more than 5 hours cumulative time spent on these problems. You may take as many breaks as you like, not counted against the 5 hours. **You should not be consulting references, working on the problems, or discussing with others during the breaks**.

Resources: The midterm and final are **not collaborative**. All questions must be done on your own, without consulting anyone else. You may consult your own notes (both in-class and notes on this class you or a colleague in the class have made), the textbook by Griffiths, and solution sets on the course website. **You may not consult any other material**, including other textbooks, the web (except for the current Phys. 402 website), material from previous years' Phys. 402 or any other classes, or copies you have made of such material, or any other resources. Calculators and symbolic manipulation programs are not allowed.

1. **Falling conducting ring**. Let's model the Earth as a perfect sphere, and model its magnetic field (above the ground) as a pure magnetic dipole given by moment $\boldsymbol{m}_E$, located at the center of the sphere.

   (a) Write down a coordinate-independent expression for this $\boldsymbol{B}$ field.

   Solution: The spherical coordinate result for the $\boldsymbol{B}$ field generated by a dipole m aligned with the $\hat{\boldsymbol{z}}$ axis is computed in Griffiths. In Eq. (5.86) he gives

$$\boldsymbol{B}_{\text{dip}} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}})\,. \tag{1}$$

   From here you can use some trigonometry to go to the coordinate-independent expression. This exercise is Griffiths' problem 5.33. The result is

$$\boldsymbol{B}_E = \frac{\mu_0}{4\pi r^3}[3(\boldsymbol{m}_E \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}_E]\,. \tag{2}$$

Now suppose we have a conducting circular ring of radius s and conductivity σ. The ring is made of wire whose cross-sectional area is A.

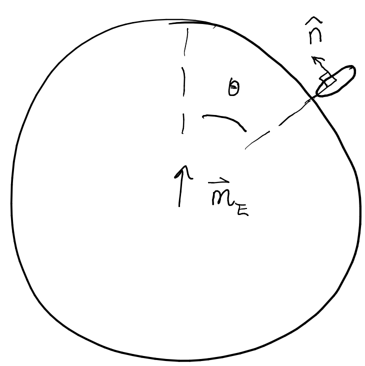
   (b) In terms of the given quantities, what is the ring's total resistance R?

   Solution: If the difference in EMF from one end to the other is V, then there is an electric field of magnitude $E = V/\ell$, where $\ell = 2\pi s$. From "Ohm's law" in a conductor, $J = \sigma E = \sigma V/\ell$. The total current through the whole cross-section is $I = JA = (A\sigma/\ell)V$. Rearranging to put this in the form of the more common Ohm's law $V = IR$, we have $R = \ell/A\sigma = 2\pi s/A\sigma$.

Now let's say we're standing on the Earth at a colatitude θ measured from the magnetic north pole. We're holding this ring so that its symmetry axis is parallel to the ground, and is pointing toward magnetic north. We drop it from distance r from the center of the Earth.

   (c) Find the magnetic flux Φ as a function of r and θ.

   Solution: Here is a diagram of the geometry of the situation:



   Since the ring is very small, we can approximate the $\boldsymbol{B}$ field as constant across the location of the ring. So, we will approximate:

$$\Phi = \int_{\mathcal{S}} \boldsymbol{B} \cdot d\boldsymbol{a} \approx \pi s^2 \hat{\boldsymbol{n}} \cdot \boldsymbol{B}_E(r,\theta) \tag{3}$$

$$= \frac{\pi s^2 \mu_0}{4\pi r^3}[3(\boldsymbol{m}_E \cdot \hat{\boldsymbol{r}})(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}}) - \hat{\boldsymbol{n}} \cdot \boldsymbol{m}_E] \tag{4}$$

$$\Phi = -\frac{s^2\mu_0}{4r^3}\hat{\boldsymbol{n}} \cdot \boldsymbol{m}_E = -\frac{s^2\mu_0}{4r^3} m_E \sin\theta\,. \tag{5}$$

   Keep in mind that Φ depends on the orientation of the surface $\mathcal{S}$. We have chosen the orientation by the direction of the $\hat{\boldsymbol{n}}$ vector which is pointing North.

(d) Assume the motion of the ring is dominated by the force of gravity, and find the EMF through the ring.

Solution: The EMF due to the motion of the ring is

$$\mathcal{E} = -\frac{d}{dt}\Phi = \frac{s^2\mu_0 m_E}{4}\frac{d}{dt}\frac{\sin\theta}{r^3}\,. \tag{6}$$

To leading approximation, the colatitude and orientation of the ring will not change. So, this is approximately

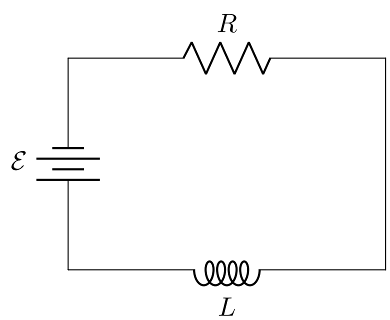
$$\mathcal{E} = \frac{s^2\mu_0 m_E \sin\theta}{4}\frac{-1}{r^4}\frac{dr}{dt} \approx \frac{s^2\mu_0 m_E \sin\theta}{4}\frac{gt}{R_E^4}\,. \tag{7}$$

In the last approximation, dr/dt was approximated by the motion being dominated by gravity. The ring will soon hit the ground, so we model the vertical velocity as if it is uniformly accelerated with the surface gravity $g \approx 9.8\ \mathrm{m/s^2}$. So, here we approximated $dr/dt \approx -gt$, for short times t, and in the denominator, $r \approx R_E$ the radius of the Earth.

Keep in mind that $\mathcal{E}$ also depends on the orientation of $\mathcal{S}$ or equivalently the path $\mathcal{P} = \partial\mathcal{S}$. This orientation is chosen to be consistent with the right-hand rule, i.e. if your right thumb points in the direction $\hat{\boldsymbol{n}}$, this is the EMF around the loop that your fingers are curling.

Suppose the ring has a self-inductance L. Now you can model the ring as a circuit with a source of EMF, a resistor, and an inductor.

(e) Draw the circuit diagram.



Solution:

(f) Write down a differential equation for the current as a function of time. Solve for $I(t)$ (assuming the ring will only fall a short distance, not through the Earth!). From the point of view of an observer looking at the ring from a point closer to the magnetic north pole, is the current flowing clockwise or anti-clockwise?

Solution: The diffeq we want to solve is

$$\mathcal{E} = IR + L\frac{dI}{dt}\,, \tag{8}$$

where $\mathcal{E} = Ct$. From Eq. (7) the constant C is

$$C = \frac{s^2\mu_0 m_E g \sin\theta}{4R_E^4}\,. \tag{9}$$

Then solving the diffeq, with the initial condition $I(0) = 0$, the solution is

$$I(t) = \frac{C\tau}{R}\left[t/\tau - 1 + \exp(-t/\tau)\right]\,, \tag{10}$$

where $\tau = L/R$. Note that this current is in the direction that your right hand's fingers are curling if your right thumb points in the direction $\hat{\boldsymbol{n}}$. The quantity in square brackets is positive, and so is the prefactor $C\tau/R$, so this is a positive current in said direction. An observer who is closer to the North pole will report the current is anti-clockwise.

Now that there is a current flowing in the ring, we can model it as a small magnetic dipole.

(g) What is its dipole moment $\boldsymbol{m}_{\text{ring}}$?

Solution: $\boldsymbol{m}_{\text{ring}} = I(t)\boldsymbol{a} \approx \pi s^2 I(t)\hat{\boldsymbol{n}} = -\pi s^2 I(t)\hat{\boldsymbol{\theta}}$, at least until it starts to rotate.

(h) What is the torque (both magnitude and direction) on the ring? Describe which way it will rotate (a diagram may be helpful).

Solution: The torque on a magnetic dipole in an external $\boldsymbol{B}$ field is

$$\boldsymbol{N} = \boldsymbol{m} \times \boldsymbol{B}\,. \tag{11}$$

Here we want to evaluate $\boldsymbol{N} = \boldsymbol{m}_{\text{ring}} \times \boldsymbol{B}_E$. Now it is actually easier to use $\boldsymbol{B}_E$ in polar coordinates from Eq. (1), since $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{\theta}}$. The cross product $-\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\theta}}$ vanishes. The only contribution comes from the part proportional to $-\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{r}} = \hat{\boldsymbol{\phi}}$. So, the torque is

$$\hat{\boldsymbol{N}} = \pi s^2 I(t) \frac{2\mu_0 m_E \cos\theta}{4\pi R_E^3} \hat{\boldsymbol{\phi}}\,. \tag{12}$$

Notice that $\cos\theta$ is positive in the Northern hemisphere and negative in the South. The ring's normal $\hat{\boldsymbol{n}}$ will be torqued to rotate around $\hat{\boldsymbol{\phi}}$ positively in the North, and negatively in the South, i.e. it will try to align with $\boldsymbol{m}_E$.

(i) What is the magnetic force on the ring (magnitude and direction)?

Solution: Griffiths writes the magnetic force on a dipole as

$$\boldsymbol{F} = \nabla(\boldsymbol{m} \cdot \boldsymbol{B})\,. \tag{13}$$

However this is actually assuming your $\boldsymbol{m}$ is independent of position. Notice that our $\boldsymbol{m}_{\text{ring}}$ implicitly depends on r and θ since $C \propto \sin\theta / r^4$. We must *not* take the derivative of anything in m when computing the force – force should be proportional to $\boldsymbol{m}$ (review the derivation of the force, or Prob. 6.22, to check). The way Griffiths writes this is just for notational convenience. To make this clearer, we should really write magnetic force as

$$F_i = m^j \nabla_i B_j\,. \tag{14}$$

To compute $\nabla_i B_j$ let's write the index notation for B_j (replacing all cases of $\hat{\boldsymbol{r}}$ with $\boldsymbol{r}/r$). Here we drop the subscript E from $\boldsymbol{m}_E$ to avoid confusion.

$$B_j = \frac{\mu_0}{4\pi r^5} 3 m_k r_k r_j - \frac{\mu_0}{4\pi r^3} m_j\,. \tag{15}$$

Now we take the ∇_i derivative, recalling that $\nabla_i r_j = \delta_{ij}$, and $\nabla_i r = \hat{r}_i = r_i/r$. After using the product and chain rules and the preceding derivatives, we get

$$\nabla_i B_j = \frac{\mu_0}{4\pi} 3 m_k \left[\frac{-5}{r^6}\frac{r_i}{r} r_k r_j + \frac{1}{r^5}\delta_{ik} r_j + \frac{1}{r^5} r_k \delta_{ij} \right] - \frac{\mu_0}{4\pi} m_j \frac{-3}{r^4}\frac{r_i}{r}\,. \tag{16}$$

As a sanity check, note that every term here goes as $1/r^4$. Now expand, contract the deltas (e.g. $m_k \delta_{ik} = m_i$), to get

$$\nabla_i B_j = \frac{\mu_0}{4\pi r^4} \left[-15(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{r}_i \hat{r}_j + 3(m_i \hat{r}_j + m_j \hat{r}_i) + 3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\delta_{ij} \right]\,. \tag{17}$$

As a second sanity check, note that if we take the trace, $\delta^{ij}\nabla_i B_j = \nabla \cdot \boldsymbol{B}$ we get the divergence of $\boldsymbol{B}$, which must vanish. Indeed it is easy to check that the trace of the above vanishes.

Now finally contracting with $\boldsymbol{m}_{\text{ring}}$, we would get

$$F_i = \frac{\mu_0}{4\pi r^4} \left[-15(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{r}_i (\hat{r}_j m^j_{\text{ring}}) + 3(m_i(\hat{r}_j m^j_{\text{ring}}) + (m_j m^j_{\text{ring}})\hat{r}_i) + 3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}}) m^i_{\text{ring}} \right]\,. \tag{18}$$

Or, going back to vector notation and replacing the E subscript on $\boldsymbol{m}_E$,

$$\boldsymbol{F} = \frac{\mu_0}{4\pi r^4}\left[-15(\boldsymbol{m}_E\cdot\hat{\boldsymbol{r}})(\hat{\boldsymbol{r}}\cdot\boldsymbol{m}_{\rm ring})\hat{\boldsymbol{r}} + 3(\boldsymbol{m}_E(\hat{\boldsymbol{r}}\cdot\boldsymbol{m}_{\rm ring}) + (\boldsymbol{m}_E\cdot\boldsymbol{m}_{\rm ring})\hat{\boldsymbol{r}}) + 3(\boldsymbol{m}_E\cdot\hat{\boldsymbol{r}})\boldsymbol{m}_{\rm ring}\right]. \tag{19}$$

The dot products we need are $\boldsymbol{m}_E\cdot\hat{\boldsymbol{r}} = m_E\cos\theta$, $\boldsymbol{m}_{\rm ring}\cdot\hat{\boldsymbol{r}} = 0$, and $\boldsymbol{m}_E\cdot\boldsymbol{m}_{\rm ring} = m_E m_{\rm ring}\sin\theta$. Plugging these in we get

$$\boldsymbol{F} = \frac{\mu_0}{4\pi r^4}\left[3(\boldsymbol{m}_E\cdot\boldsymbol{m}_{\rm ring})\hat{\boldsymbol{r}} + 3(\boldsymbol{m}_E\cdot\hat{\boldsymbol{r}})\boldsymbol{m}_{\rm ring}\right] \tag{20}$$

$$\boldsymbol{F} = \frac{3\mu_0 m_E m_{\rm ring}}{4\pi r^4}\left[\sin\theta\hat{\boldsymbol{r}} - \cos\theta\hat{\boldsymbol{\theta}}\right]. \tag{21}$$

2. **Magnetic field between two coils**. We have two circular current loops in our lab, each of radius R. We place them both on the z axis with their symmetry axes also along z. One of them is at height $z = +a$, and the other is at height $z = -a$. We run the same current I through both coils.

(a) Write down an integral for the mutual inductance between the two loops. (This should be an ordinary one-dimensional integral; make as much progress as possible but you may not be able to completely evaluate it)

Solution: Starting from the Neumann formula,

$$M = \frac{\mu_0}{4\pi}\oint_{L_1}\oint_{L_2}\frac{d\boldsymbol{\ell}_1\cdot d\boldsymbol{\ell}_2}{\mathscr{r}}, \tag{22}$$

where L_1 and L_2 are loops 1 and 2. Parameterize each loop by an angle θ_1 and θ_2 where this is the angle in the $x-y$ plane measured from the x axis. The arc length is $d\boldsymbol{\ell}_1 = Rd\theta_1\hat{\boldsymbol{n}}_1$ and similarly for loop 2. The dot product between the two direction vectors is $\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2 = \cos(\theta_1-\theta_2)$. The separation $\mathscr{r}$ can be computed from the vertical distance, $2a$, and the distance in the $x-y$ plane, which we will call p, as $\mathscr{r}^2 = (2a)^2 + p^2$. To compute p you can use the law of cosines for an isosceles triangle going from the origin to the points (R,θ_1) and (R,θ_2) in polar coordinates (the opening angle at the origin is then $|\theta_1-\theta_2|$). From law of cosines we get $p^2 = R^2 + R^2 - 2R^2\cos(\theta_1-\theta_2)$. Combining we have

$$\mathscr{r}^2 = 4a^2 + 2R^2(1-\cos(\theta_1-\theta_2)). \tag{23}$$

Put this into the Neumann formula, which you see depends only on the difference $\theta_1-\theta_2$. That means you can change one integration variable to $\theta' = \theta_1-\theta_2$, and then do the other integration analytically, resulting in a single integral,

$$M = \frac{\mu_0}{2}\int_0^{2\pi}\frac{R^2\cos\theta' d\theta'}{\sqrt{4a^2+2R^2(1-\cos\theta')}} \tag{24}$$

$$M = \frac{\mu_0 R}{2}\int_0^{2\pi}\frac{\cos\theta' d\theta'}{\sqrt{4a^2/R^2+2(1-\cos\theta')}}. \tag{25}$$

This integral can in fact be represented in terms of special functions called "complete elliptic integrals," but that is not necessary here.

(b) What is the magnetic field along the z axis, $\boldsymbol{B}(\rho=0,z)$, in the region between the two loops? Show that $\left.\frac{\partial B_z}{\partial z}\right|_{z=0} = 0$.

Solution: Because of linearity, we will be able to superpose the magnetic fields of each individual loop. Consider an individual loop of radius R, lying in the $x-y$ plane. We can get the magnetic field from the Biot-Savart law,

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}I\oint\frac{d\boldsymbol{\ell}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}. \tag{26}$$

On the z axis, because of azimuthal symmetry, the components of $\boldsymbol{B}$ in the $x-y$ plane must add up to 0, so the only component we need to compute is B_z. Again from azimuthal symmetry, the contribution from every infinitesimal of the loop will be the same, so we just need 2π times the infinitesimal contribution at one point.

Consider contribution to $B_z(z)$ from the element of wire on the x axis. The separation is $\mathscr{r}^2 = R^2 + z^2$. $d\boldsymbol{\ell}'$ points in the $\hat{y}$ direction, and $\hat{\boldsymbol{\mathscr{r}}}$ is a unit vector pointing from $(R, 0, 0)$ to $0, 0, z$. Computing the z component of the cross product gives $(d\boldsymbol{\ell}' \times \hat{\boldsymbol{\mathscr{r}}})_z = dl' \cos\theta$ where $\tan\theta = z/R$. So we would find

$$B_z^{\text{single}}(z) = \frac{\mu_0 I}{4\pi} \int \frac{\cos\theta dl'}{R^2 + z^2} = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \,. \tag{27}$$

This is positive both above and below the plane of the loop.

Now we superpose two of these solutions, putting one loop at $z = -a$, and the other at $z = +a$. The result is

$$B_z(z) = \frac{\mu_0 I R^2}{2} \left[\frac{1}{[R^2 + (z-a)^2]^{3/2}} + \frac{1}{[R^2 + (z+a)^2]^{3/2}} \right] . \tag{28}$$

The z derivative is

$$\frac{\partial B_z}{\partial z} = \frac{\mu_0 I R^2}{2} \left[\frac{-\frac{3}{2} 2(z-a)}{[R^2 + (z-a)^2]^{5/2}} + \frac{-\frac{3}{2} 2(z+a)}{[R^2 + (z+a)^2]^{5/2}} \right] . \tag{29}$$

Evaluating at $z = 0$ there is a cancellation, $\partial B_z / \partial z|_{z=0} = 0$.

(c) Find the value of a that will make the z component of the magnetic field more uniform at the center, so as to satisfy

$$\left. \frac{\partial^2 B_z}{\partial z^2} \right|_{z=0} = 0 \,.$$

Solution: Compute the second derivative,

$$\frac{\partial^2 B_z}{\partial z^2} = \frac{\mu_0 I R^2}{2} \left[\frac{15(z-a)^2}{[R^2 + (z-a)^2]^{7/2}} + \frac{15(z+a)^2}{[R^2 + (z+a)^2]^{7/2}} + \frac{-3}{[R^2 + (z-a)^2]^{5/2}} + \frac{-3}{[R^2 + (z+a)^2]^{5/2}} \right] . \tag{30}$$

Evaluate this at $z = 0$ to find

$$\left. \frac{\partial^2 B_z}{\partial z^2} \right|_{z=0} = \frac{\mu_0 I R^2}{2} \left[\frac{30a^2}{(R^2 + a^2)^{7/2}} + \frac{-6}{(R^2 + a^2)^{5/2}} \right] . \tag{31}$$

We want this to vanish, so we solve for a in this equation,

$$0 = \frac{30a^2}{(R^2 + a^2)^{7/2}} + \frac{-6}{(R^2 + a^2)^{5/2}} \tag{32}$$

$$0 = 30a^2 - 6(R^2 + a^2) \,. \tag{33}$$

This quadratic is easily solved with the two solutions being $a = \pm\frac{R}{2}$, which is really the same solution. This configuration is known as a *Helmholtz coil.*

3. **Taking a few derivatives**. Suppose we create an electromagnetic field given by

$$\boldsymbol{E} = (\beta \sin y \,\hat{\boldsymbol{x}} + \beta \cos x \,\hat{\boldsymbol{y}} + \alpha \sin z \,\hat{\boldsymbol{z}}) e^{-t} \,,$$
$$\boldsymbol{B} = -\beta(\cos y + \sin x) e^{-t} \,\hat{\boldsymbol{z}} \,.$$

(a) Show that $\nabla \cdot \boldsymbol{B} = 0$ is satisfied.

Solution: Only nonvanishing component is B_z, so only need $\partial B_z / \partial z$, which is trivially zero.

(b) Show that $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ is satisfied.
Solution:

$$\nabla \times \boldsymbol{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{\boldsymbol{x}} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{\boldsymbol{y}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{\boldsymbol{z}} \tag{34}$$

$$= -\beta e^{-t} (\sin x + \cos y)\, \hat{\boldsymbol{z}} \tag{35}$$

$$= -\frac{\partial \boldsymbol{B}}{\partial t} \tag{36}$$

(c) What charge density ρ and current density $\boldsymbol{J}$ would produce this electromagnetic field?
Solution: Notice that only E_z contributes to the divergence below.

$$\rho = \epsilon_0 \nabla \cdot \boldsymbol{E} = \epsilon_0 \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \tag{37}$$

$$= \epsilon_0 \alpha \cos z e^{-t} \tag{38}$$

$$\boldsymbol{J} = \frac{1}{\mu_0} \left[\nabla \times \boldsymbol{B} - \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right] \tag{39}$$

$$= \left[\frac{\beta(1 + \epsilon_0 \mu_0)}{\mu_0} (\sin y \hat{\boldsymbol{x}} + \cos x \hat{\boldsymbol{y}}) + \epsilon_0 \alpha \sin z \hat{\boldsymbol{z}} \right] e^{-t} \tag{40}$$

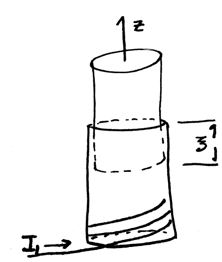
(d) Show that $\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0$ is satisfied.
Solution: Here only J_z contributes to the divergence so it's again simple,

$$\nabla \cdot \boldsymbol{J} = \epsilon_0 \alpha e^{-t} \cos z = -\frac{\partial \rho}{\partial t}\,. \tag{41}$$

(e) Compute the Poynting vector $\boldsymbol{S}$ of this field configuration.
Solution:

$$\boldsymbol{S} = \mu_0 \boldsymbol{E} \times \boldsymbol{B} = \frac{\beta^2 e^{-2t} (\cos y + \sin x)}{\mu_0} (-\cos x \hat{\boldsymbol{x}} + \sin y \hat{\boldsymbol{y}}) \tag{42}$$

4. **Partially overlapping solenoids**. Suppose we have two almost-identical solenoids: they have length ℓ, with N total turns, giving $n \equiv N/\ell$ turns per unit length, and the wire in each solenoid has a total resistance R. One of them has radius $r + \epsilon$ and the other has radius r, so they can just barely nest within each other (but for calculation purposes we can consider both of them to have radius r).
We place both of these solenoids along the z axis, with a length ξ of overlap (see figure). However this overlap is not fixed – the two solenoids can move in response to forces, and thus $\xi(t)$ may depend on time.



Compute the mutual inductance $M(\xi)$ of the two solenoids, which will depend on ξ (you can assume that when you run current through one of the solenoids, it will create a uniform magnetic field inside and no magnetic field outside).

Solution: Suppose we turn on current I_1 through solenoid 1, which will create a magnetic field $\boldsymbol{B} = \mu_0 n I_1 \hat{\boldsymbol{z}}$ in the interior. Now consider an individual loop of solenoid 2, which overlaps with the interior of solenoid 1. The flux through a single loop will be $\Phi_{\text{loop}} = \int \boldsymbol{B} \cdot d\boldsymbol{a} = \pi r^2 \mu_0 n I_1$. The total flux in circuit 2 is due just to those loops (there are $n\xi(t)$ of them) of the solenoid 2 that overlap with solenoid 1. Therefore we have

$$\Phi_2 = \Phi_{\text{loop}}(n\xi(t)) = \pi r^2 \mu_0 n I_1 n \xi(t)\,, \tag{43}$$

$$M = M(\xi) = \pi r^2 \mu_0 n^2 \xi(t)\,, \tag{44}$$

where in the last line all we did is find the coefficient in $\Phi_2 = M I_1$. Importantly, note that the mutual inductance is a function of ξ and thus can be a function of time.

5. **Magnetic fluid in a solenoid**. Let's take a solenoid of radius r and height h, with $n \equiv N/h$ turns per unit length. We hold this solenoid vertically just on the surface of a bath of incompressible fluid of density ρ. This fluid is a linear magnetic medium with magnetic susceptibility χ_m, either positive or negative. Suppose that at some instant, this fluid also fills the solenoid up to height ζ (i.e. the inside of the solenoid has fluid in the range $0 \le z \le \zeta$, and it has air in the range $\zeta \le z \le h$).

   (a) What is the magnetic field $\boldsymbol{B}$ in each of the two regions (fluid and air)?

   Solution: Recall the solution for the magnetic field in a solenoid filled with a linear medium of given susceptibility (Griffiths' example 6.3): the field inside is $B = \mu n I = \mu_0(1+\chi_m)nI$ for the appropriate value of χ_m. In other words, we simply replace μ_0 with $\mu = \mu_0(1+\chi_m)$ in the fluid. So, the complete field is

$$\boldsymbol{B} = \hat{\boldsymbol{z}} \begin{cases} \mu_0 n I\,, & \zeta \le z \le h \\ \mu n I\,, & 0 \le z \le \zeta\,. \end{cases} \tag{45}$$

   (b) Find the total self-inductance L of the partially-filled solenoid, as a function of ζ.

   Solution: Let's treat the fluid-filled and air-filled parts of the solenoid as two separate solenoids that are in series with each other. If the two separate parts have self-inductances L_{fluid} and L_{air}, the total self-inductance will be $L = L_{\text{fluid}} + L_{\text{air}}$.

   Now, the self-inductance of the air (or vacuum) part was computed as an exercise in the book. Let's consider the fluid-filled part. The flux through one winding will be $\Phi_{\text{one}} = \pi r^2 \mu n I$, so the total flux is $\Phi = (\pi r^2 \mu n I)(\zeta n)$, since there are ζn windings in this part. This gives a self-inductance of $L_{\text{fluid}} = \pi r^2 \mu n^2 \zeta$. Similarly, the self-inductance of the air part is $L_{\text{air}} = \pi r^2 \mu_0 n^2 (h-\zeta)$. This gives a total inductance of

$$L = \pi r^2 n^2 (\mu\zeta + \mu_0(h-\zeta)) = \pi r^2 n^2 \mu_0 (h + \zeta\chi_m)\,. \tag{46}$$

   (c) Find the energy stored in the magnetic field as a function of ζ. There are two (or more) ways to do this – check the correctness of your result by showing both calculations.

   Solution: One approach is to use the energy stored in an inductor,

$$E_1 = \frac{1}{2} L I^2 = \frac{I^2}{2} \pi r^2 n^2 \mu_0 (h + \zeta\chi_m) \tag{47}$$

   On the other hand, a second approach is to use the volume integral of the energy density in the magnetic field plus the energy necessary to polarize the material (if present). Recall that for a linear medium, this works out to

$$E_2 = \int_{\mathcal{V}} \frac{1}{2\mu} B^2 \, d^3\text{Vol}\,, \tag{48}$$

   where we use μ_0 in the vacuum (or air) region, and use $\mu = \mu_0(1+\chi_m)$ in the fluid region. Since B^2 is roughly constant in each region, and the B field is very small outside, we approximate this as

$$E_2 = \frac{1}{2\mu_0} B_{\text{air}}^2 V_{\text{air}} + \frac{1}{2\mu_0(1+\chi_m)} B_{\text{fluid}}^2 V_{\text{air}}\,, \tag{49}$$

   where the respective volumes are $V_{\text{fluid}} = \pi r^2 \zeta$ and $V_{\text{fluid}} = \pi r^2 (h-\zeta)$. Combining everything we see that the two approaches give the same energy, $E_1 = E_2$.

   (d) If we turn on a current in the solenoid, the fluid may be pulled upward, depending on the sign of the susceptibility. Will a positive or negative χ_m be pulled upward?

   Solution: The fluid will be pulled upward if that would *decrease* the energy, so we want the coefficient of ζ to be negative. This means the fluid will be pulled upward for $\chi_m < 0$, i.e. for a diamagnetic material.

(e) There is an energy cost to raising this fluid to height ζ. Write the total energy (gravitational and magnetic). Find the equilibrium height $\zeta_{\rm eq}$ which the fluid achieves as a function of the current I (and any other quantities in the problem).

Solution: The gravitational potential energy (density) of each element of fluid of density ρ at a height z is $\epsilon = \rho g z$. If the fluid extends between $0 \le z \le \zeta$, then the total gravitational potential energy will be

$$E_g = \int_0^\zeta \pi r^2 \, \rho g z \, dz = \pi r^2 \rho g \frac{\zeta^2}{2} \, . \tag{50}$$

So, the total energy as a function of fluid height ζ is

$$E(\zeta) = \frac{I^2}{2} \pi r^2 n^2 \mu_0 (h + \zeta \chi_m) + \pi r^2 \rho g \frac{\zeta^2}{2} \, . \tag{51}$$

The equilibrium value $\zeta_{\rm eq}$ will be at the minimum of the energy, where

$$E'(\zeta) = \pi r^2 (\rho g \zeta + I^2 n^2 \mu_0 \chi_m) \, , \tag{52}$$

$$E'(\zeta_{\rm eq}) = 0 \, . \tag{53}$$

Solving for $\zeta_{\rm eq}$ we find

$$\zeta_{\rm eq} = -\chi_m \frac{I^2 n^2 \mu_0}{2 g \rho} \, , \tag{54}$$

as long as this quantity ends up in the range $0 \le \zeta_{\rm eq} \le h$ (notice that $\zeta_{\rm eq}$ is positive when χ_m is negative, as found above). This is the case so long as $I^2 \le 2\rho g h / (\mu_0 n^2 \chi_m)$.

6. **Standing electromagnetic wave**. Suppose we superimpose the following two complex, monochromatic waves, both with frequency ω: one traveling in the $\hat{z}$ direction, and the electric field is polarized in $\hat{x}$ with amplitude $\tilde{E}_0$; and the second is traveling in the $-\hat{z}$ direction, with electric field polarized in $-\hat{x}$ with the same amplitude $\tilde{E}_0$.

(a) Write down the complex $\tilde{\boldsymbol{E}}, \tilde{\boldsymbol{B}}$ fields for each of the two waves separately.

Solution: We will denote the left-moving and right-moving fields as $\tilde{\boldsymbol{E}}_L, \tilde{\boldsymbol{E}}_R$, etc.

$$\tilde{\boldsymbol{E}}_R = +\tilde{E}_0 \exp[i(kz - \omega t)] \hat{\boldsymbol{x}} \, , \tag{55}$$

$$\tilde{\boldsymbol{B}}_R = +\frac{1}{c} \tilde{E}_0 \exp[i(kz - \omega t)] \hat{\boldsymbol{y}} \, , \tag{56}$$

$$\tilde{\boldsymbol{E}}_L = -\tilde{E}_0 \exp[i(-kz - \omega t)] \hat{\boldsymbol{x}} \, , \tag{57}$$

$$\tilde{\boldsymbol{B}}_L = +\frac{1}{c} \tilde{E}_0 \exp[i(-kz - \omega t)] \hat{\boldsymbol{y}} \, . \tag{58}$$

(b) Now superimpose them and find the *real* field that arises from the superposition.

Solution: The physical fields are $\boldsymbol{E} = \mathrm{Re}[\tilde{\boldsymbol{E}}_L + \tilde{\boldsymbol{E}}_R]$ and $\boldsymbol{B} = \mathrm{Re}[\tilde{\boldsymbol{B}}_L + \tilde{\boldsymbol{B}}_R]$. This gives

$$\boldsymbol{E} = 2 E_0 \sin(kz) \sin(\omega t) \hat{\boldsymbol{x}} \, , \tag{59}$$

$$\boldsymbol{B} = \frac{2}{c} E_0 \cos(kz) \cos(\omega t) \hat{\boldsymbol{y}} \, . \tag{60}$$

(c) Find the part of the period-averaged energy density due just to the electric field and comment on its pattern in space.

Solution: We are going to compute $\langle u_E \rangle = \langle \frac{1}{2} \epsilon_0 E^2 \rangle$,

$$\langle \frac{1}{2} \epsilon_0 E^2 \rangle = \frac{1}{T} \int_0^T \frac{1}{2} \epsilon_0 (2 E_0 \sin(kz) \sin(\omega t))^2 dt \tag{61}$$

$$\langle \frac{1}{2} \epsilon_0 E^2 \rangle = \epsilon_0 E_0^2 \sin^2(kz) \, , \tag{62}$$

where the period is $T = 2\pi/\omega$. Notice that this is spatially modulated.

(d) Now examine the total period-averaged energy density $\langle u \rangle$ (due to both the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields), and again comment on its pattern in space.

Solution: Repeating for the magnetic field we find $\langle u_B \rangle = \epsilon_0 E_0^2 \cos^2(kz)$, of the same amplitude and also spatially modulated, but with a different phase. Thus the sum is not spatially modulated: $\langle u \rangle = \epsilon_0 E_0^2$.

(e) Finally, find the period-averaged Poynting vector, $\langle \boldsymbol{S} \rangle$.

Solution: Computing the real Poynting vector we find

$$\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B} = \frac{4}{\mu_0 c} E_0^2 \sin(kz) \sin(\omega t) \cos(kz) \cos(\omega t) \hat{\boldsymbol{z}} \,. \tag{63}$$

Now when we period average we need the integral

$$\frac{1}{T} \int_0^T \sin(\omega t) \cos(\omega t) dt = 0 \,. \tag{64}$$

Therefore $\langle \boldsymbol{S} \rangle = 0$, and there is no energy flux.