UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2020

Problem Set 4 — SOLUTIONS

Due: Wednesday, Mar. 4, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. Griffiths problem 9.2 (Standing waves are superposed traveling waves).

   Solution: Suppose we have the standing wave $f(z,t) = A\sin(kz)\cos(kvt)$. Check that it solves the wave equation:

$$\frac{\partial f}{\partial z} = kA\cos(kz)\cos(kvt) \tag{1}$$
$$\frac{\partial^2 f}{\partial z^2} = -k^2 A\sin(kz)\cos(kvt) = -k^2 f \tag{2}$$
$$\frac{\partial f}{\partial t} = -kvA\sin(kz)\sin(kvt) \tag{3}$$
$$\frac{\partial^2 f}{\partial t^2} = -(kv)^2 A\sin(kz)\cos(kvt) = -(kv)^2 f\,. \tag{4}$$

   Thus evaluating the wave equation

$$\frac{-1}{v^2}\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}(kv)^2 f - k^2 f = 0 \quad \checkmark \tag{5}$$

   Next representing this as a superposition of left and right traveling waves. All we need is the trig addition identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. You can immediately verify that

$$f(z,t) = A\sin(kz)\cos(kvt) = \frac{A}{2}\left[\sin(kz - kvt) + \sin(kz + kvt)\right]\,. \tag{6}$$

2. Griffiths problem 9.5 (Wave incident on a boundary where two materials meet).

   Solution: We will impose two boundary conditions at the interface $z = 0$: first, $f(0^-,t) = f(0^+,t)$, and second, $\frac{\partial f}{\partial z}\big|_{0^-} = \frac{\partial f}{\partial z}\big|_{0^+}$. Our functions to the left and right are:

$$f(z,t) = \begin{cases} g_I(z - v_1 t) + h_R(z + v_1 t) & z < 0 \\ g_T(z - v_2 t) & z > 0\,. \end{cases} \tag{7}$$

   The first B.C. we impose is $f(t,0^-) = f(t,0^+)$. This tells us that

$$g_I(-v_1 t) + h_R(+v_1 t) = g_T(-v_2 t) \tag{8}$$
$$g_I(w) + h_R(-w) = g_T(\frac{v_2}{v_1}w)\,, \tag{9}$$

   where we have defined $w \equiv v_1 t$. The second B.C. we impose is

$$\left.\frac{\partial f}{\partial z}\right|_{0^+} = \left.\frac{\partial f}{\partial z}\right|_{0^+} \tag{10}$$
$$g_I'(w) + h_R'(-w) = g_T'(\frac{v_2}{v_1}w)\,. \tag{11}$$

Let us integrate this with respect to w. The functions h'_R and g'_T will need variable substitutions to perform the integrals, giving

$$g_I(w) - h_R(-w) = \frac{v_1}{v_2} g_T(\frac{v_2}{v_1} w) + C_1 \,, \tag{12}$$

with some integration constant C_1 that depends on initial conditions. The two equations (9) and (12) form a linear system for $h_R(-w)$ and $g_T(v_2 w/v_1)$. Add the two equations and change w to $u = v_2 w/v_1$ to find

$$g_T(u) = \frac{2v_2}{v_1 + v_2} g_I(\frac{v_1}{v_2} u) + C_2 \,, \tag{13}$$

where $C_2 = -C_1 v_2/(v_1 + v_2)$. Now eliminating g_T from the system, we solve for $h_R(-w)$ (and now present it as a function of $u = -w$),

$$h_R(w) = \frac{v_2 - v_1}{v_1 + v_2} g_I(-w) + C_2 \,. \tag{14}$$

3. Griffiths problem 9.8 (Circularly polarized wave).

   Solution: We are working with the wave

$$\tilde{\boldsymbol{f}} = \tilde{A} e^{i(kz - \omega t)} \left(e^{i\delta_v} \hat{\boldsymbol{x}} + e^{i\delta_h} \hat{\boldsymbol{y}} \right) \tag{15}$$

   where we have set the $\hat{\boldsymbol{x}}$ direction to be vertical, and we are using $\delta_v = 0, \delta_h = \pi/2$, so

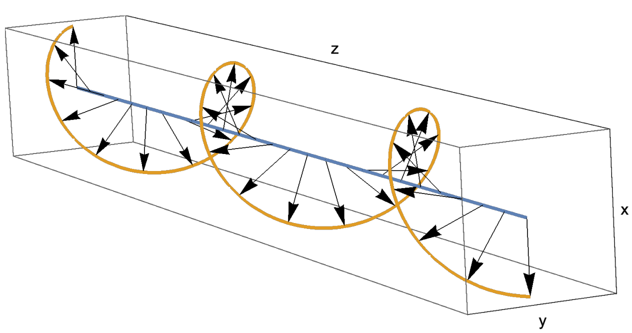
$$\tilde{\boldsymbol{f}} = \tilde{A} e^{i(kz - \omega t)} \left(\hat{\boldsymbol{x}} + i \hat{\boldsymbol{y}} \right) \,. \tag{16}$$

   (a) The physical string motion is given by $\boldsymbol{f} = \mathrm{Re}[\hat{\boldsymbol{f}}]$. Without loss of generality, let's choose $\tilde{A} = A$ to be real, and take $z = 0$. The string motion will be

$$\boldsymbol{f} = A\, \mathrm{Re}[e^{-i\omega t} \left(\hat{\boldsymbol{x}} + i \hat{\boldsymbol{y}} \right)] \tag{17}$$

$$= A \left[\cos(-\omega t) \hat{\boldsymbol{x}} + \cos(\pi/2 - \omega t) \hat{\boldsymbol{y}} \right] = A \left[\cos(\omega t) \hat{\boldsymbol{x}} + \sin(\omega t) \hat{\boldsymbol{y}} \right] \,. \tag{18}$$

   This is parametrically describing a point moving around the circle of radius A in the $x - y$ plane. At time $t = 0$, it is on the positive x axis. At time $\pi/2\omega$, it is on the positive y axis. Thus if you are at positive z and looking toward the origin, you will see the point circles **counter-clockwise**. To make it circle clockwise, set $\delta_h = -\pi/2$.

   (b) 

   (c) Just hold the end while moving your wrist or arm periodically in a steady circle.

4. Griffiths problem 9.9a-b (The real $\boldsymbol{E}$ and $\boldsymbol{B}$ fields from two example monochromatic plane waves).

   Solution:

   (a) $\boldsymbol{k} = (-\omega/c)\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$. From these we can compute $\boldsymbol{k} \cdot \boldsymbol{r} = (-\omega/c)x$ (recall that $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$). We can also compute $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}} = \hat{\boldsymbol{y}}$. Now we can write down the real fields,

$$\boldsymbol{E} = E_0 \cos(\frac{\omega}{c} x + \omega t) \hat{\boldsymbol{z}} \,, \qquad \boldsymbol{B} = \frac{E_0}{c} \cos(\frac{\omega}{c} x + \omega t) \hat{\boldsymbol{y}} \,. \tag{19}$$

(b) $\boldsymbol{k}$ should have direction $(1,1,1)$ but magnitude ω/c. The normalization constant is found very easily, and $\boldsymbol{k} = (\omega/c)(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}})/\sqrt{3}$. To find $\hat{\boldsymbol{n}}$, write it as $\hat{\boldsymbol{n}} = a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{z}}$, where $a^2 + b^2 = 1$. Now require that $\hat{\boldsymbol{n}} \cdot \boldsymbol{k} = 0$ and find $\hat{\boldsymbol{n}} = (\hat{x} - \hat{z})/\sqrt{2}$. Now we can compute $\boldsymbol{k} \cdot \boldsymbol{r} = (\omega/\sqrt{3}c)(x + y + z)$ and $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}} = (-\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}})/\sqrt{2}$. Combining we get

$$\boldsymbol{E} = E_0 \cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z) - \omega t\right](\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}})/\sqrt{2}\,, \tag{20}$$

$$\boldsymbol{B} = \frac{E_0}{c} \cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z) - \omega t\right](-\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}})/\sqrt{6}\,. \tag{21}$$

5. Griffiths problem 9.12 (The Maxwell stress tensor due to a monochromatic plane wave traveling in the z direction).

Solution: Starting from

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)\,, \tag{22}$$

we will plug in a plane wave traveling in the z direction, linearly polarized in the x direction, namely

$$\boldsymbol{E} = E_0 \cos(kz - \omega t)\hat{\boldsymbol{x}}\,, \qquad \boldsymbol{B} = \frac{E_0}{c}\cos(kz - \omega t)\hat{\boldsymbol{y}}\,. \tag{23}$$

The only non-vanishing components in this case are the diagonal ones, which take on values

$$T_{xx} = \epsilon_0 \left(E_x E_x - \frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B^2\right) = \frac{1}{2}\left(\epsilon_0 E^2 - \frac{1}{\mu_0}B^2\right) = 0\,, \tag{24}$$

$$T_{yy} = \epsilon_0 \left(-\frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(B_y B_y - \frac{1}{2}B^2\right) = \frac{1}{2}\left(-\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = 0\,, \tag{25}$$

$$T_{zz} = \epsilon_0 \left(-\frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B^2\right) = -\epsilon_0 E_0^2 \cos^2(kz - \omega t) = -u\,. \tag{26}$$

This is consistent with the fact that T_{ij} quantifies the amount of $i-$momentum being transported in the j direction. In this case we have momentum flux density = energy density.

6. **Stress in index notation** (extra credit). Suppose you have a monochromatic plane wave where the direction and k-number are given by the vector $\boldsymbol{k}$, or k_i in index notation; and this wave is linearly polarized with unit polarization vector $\hat{\boldsymbol{e}}$, or $\hat{e}_i$ in index notation, where $\boldsymbol{k} \cdot \hat{\boldsymbol{e}} = 0 = k_i \hat{e}_i$. Find the Maxwell stress tensor T_{ij} in index notation, in terms of the above quantities.

Solution: Translating the linearly-polarized plane wave solution into index notation, we have

$$E_i = \hat{e}_i E_0 \cos(k_j x^j - \omega t)\,, \tag{27}$$

$$B_i = \epsilon_{ijk} \hat{k}^j \hat{e}^k \frac{E_0}{c}\cos(k_j x^j - \omega t)\,. \tag{28}$$

(We must be careful since k is being used as both an index and the name of a vector). Now we need to insert this into

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)\,, \tag{29}$$

and make use of the identity for the product of two epsilon tensors, which can be compactly written as

$$\epsilon^{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta^i_l & \delta^i_m & \delta^i_n \\ \delta^j_l & \delta^j_m & \delta^j_n \\ \delta^k_l & \delta^k_m & \delta^k_n \end{vmatrix} = \delta^i_l\left(\delta^j_m \delta^k_n - \delta^j_n \delta^k_m\right) - \delta^i_m\left(\delta^j_l \delta^k_n - \delta^j_n \delta^k_l\right) + \delta^i_n\left(\delta^j_l \delta^k_m - \delta^j_m \delta^k_l\right)\,. \tag{30}$$

First, let's compute both E^2 and B^2. The first is very simple,

$$E^2 = E_i E^i = E_0^2 \hat{e}_i \hat{e}^i \cos^2 S = E_0^2 \cos^2 S\,, \tag{31}$$

where $S \equiv k_j x^j - \omega t$ and we have used the fact that $\hat{e}_i \hat{e}^i = \hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{e}} = 1$. Now for the more complicated B^2, where we have to make use of Eq. (30),

$$B^2 = B_i B^i = \frac{E_0^2}{c^2} \cos^2 S\, \epsilon_{ijk} \epsilon^i{}_{lm} \hat{k}^j \hat{e}^k \hat{k}^l \hat{e}^m \tag{32}$$

$$B^2 = \frac{E_0^2}{c^2} \cos^2 S\, \left((\hat{k}_j \hat{k}^j)(\hat{e}_k \hat{e}^k) - (\hat{k}_j \hat{e}^j)(\hat{k}_k \hat{e}^k) \right) \tag{33}$$

$$B^2 = \frac{E_0^2}{c^2} \cos^2 S \tag{34}$$

where have also made use of $\hat{e}_i \hat{e}^i = \hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{e}} = 1$ and $\hat{e}_i \hat{k}^i = \hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{k}} = 0$.

Finally we can assemble the Maxwell stress tensor,

$$T_{ij} = \epsilon_0 E_0^2 \cos^2 S \left[\left(\hat{e}_i \hat{e}_j - \frac{1}{2} \delta_{ij} \right) + \left(\epsilon_{ikl} \epsilon_{jmn} \hat{k}^k \hat{e}^l \hat{k}^m \hat{e}^n - \frac{1}{2} \delta_{ij} \right) \right] . \tag{35}$$

Now we need to expand out the somewhat complicated expression $\epsilon_{ikl} \epsilon_{jmn} \hat{k}^k \hat{e}^l \hat{k}^m \hat{e}^n$. After using Eq. (30), contracting the δ tensors where possible, and using $\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{e}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1$ and $\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{k}} = 0$, we arrive at

$$\epsilon_{ikl} \epsilon_{jmn} \hat{k}^k \hat{e}^l \hat{k}^m \hat{e}^n = \delta_{ij} - \hat{e}_i \hat{e}_j - \hat{k}_i \hat{k}_j\,. \tag{36}$$

Plugging this back in we finally get

$$T_{ij} = \epsilon_0 E_0^2 \cos^2 S\, k_i k_j = -u\, \hat{k}_i \hat{k}_j\,. \tag{37}$$

There is a simpler way to arrive at this result. First study the case where $\hat{k} = \hat{z}$, as in Griffiths problem 9.12. There, the choice of coordinate axes were totally arbitrary, so we must be able to write the stress tensor in a way that doesn't depend on choice. Now identify the equality of the two expressions $T_{ij} = -u \hat{k}_i \hat{k}_j$ in that problem. Since this result does not depend on choice of coordinates, we can promote the $\hat{k} = \hat{z}$ result to be valid for any $\hat{k}$. A slightly more justified approach is to transform the tensor result from the $\hat{k} = \hat{z}$ case by applying the correct rotations to transform $\hat{z}$ to point in the correct direction. This requires knowing how tensors (or at least matrices) transform under a rotation.