UNIVERSITY OF MISSISSIPPI
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Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2020

Problem Set 2 — SOLUTIONS

Due: Weds., Feb. 12, 2020, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. Griffiths problem 7.30 (Mutual inductance between two tiny wire loops).

   Solution:

   (a) Since mutual inductance is reciprocal, we can take either loop as having a sustained current, and compute the current induced in the other. Put loop 1 at the origin. If loop 1 has current I_1, it generates approximately $\boldsymbol{B}_1 = \frac{\mu_0}{4\pi}\frac{1}{r^3}I_1[3(\boldsymbol{a}_1 \cdot \hat{\boldsymbol{\mathscr{r}}})\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{a}_1]$. Then the flux through loop 2 is $\Phi_2 = \boldsymbol{B}(\boldsymbol{r} = \boldsymbol{r}_2)\cdot\boldsymbol{a}_2 = \frac{\mu_0}{4\pi}\frac{1}{r^3}I_1[3(\boldsymbol{a}_1\cdot\hat{\boldsymbol{\mathscr{r}}})(\hat{\boldsymbol{\mathscr{r}}}\cdot\boldsymbol{a_2}) - \boldsymbol{a}_1\cdot\boldsymbol{a}_2] = MI_1$. Thus $M = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\boldsymbol{a}_1\cdot\hat{\boldsymbol{\mathscr{r}}})(\hat{\boldsymbol{\mathscr{r}}}\cdot\boldsymbol{a_2}) - \boldsymbol{a}_1\cdot\boldsymbol{a}_2]$ which is clearly symmetric under the exchange $1 \leftrightarrow 2$.

   (b) Due to the changing current $I_2(t) \neq \text{const.}$, there will be an EMF in loop one, $\mathcal{E}_1 = -M\frac{d}{dt}I_2$. If a current controller on loop 1 is keeping its current at a constant I_1 then the work it's doing per unit time is $\frac{dW}{dt} = -\mathcal{E}_1 I_1 = MI_1\frac{d}{dt}I_2$. Integrate this over the time it takes to change the current in loop 2, but the whole thing is just a constant times the total derivative $\frac{d}{dt}I_2$. Thus the work that the current controller performs is $W = MI_1I_2 = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\boldsymbol{m}_1\cdot\hat{\boldsymbol{\mathscr{r}}})(\hat{\boldsymbol{\mathscr{r}}}\cdot\boldsymbol{m_2}) - \boldsymbol{m}_1\cdot\boldsymbol{m}_2]$ where $\boldsymbol{m}_1 = \boldsymbol{a}_1 I_1$ and similarly for loop 2. This is the same as Eq. (6.35) except for a minus sign. That equation only accounted for the energy in the fields, whereas here we have the energy in the source currents as well. The fact that the sum of (fields+sources) is the same as (-fields) is extremely common in linear theories (such as electromagnetism) and can be explained more deeply from a Lagrangian or Hamiltonian point of view.

2. Griffiths problem 7.53 (ratio of EMFs in a transformer).

   Solution: Consider an individual turn of the winding of either the primary or secondary coil. Suppose the amount of flux through this one turn is Φ. Then the amount of flux through the primary and secondary is (respectively) $\Phi_1 = N_1\Phi$ and $\Phi_2 = N_2\Phi$. Thus the respective EMFs are $\mathcal{E}_1 = -\frac{d}{dt}\Phi_1 = -N_1\frac{d}{dt}\Phi$ and similarly $\mathcal{E}_2 = -N_2\frac{d}{dt}\Phi$. Thus we immediately find $\mathcal{E}_1/\mathcal{E}_2 = N_1/N_2$.

3. Suppose we have two circuits – of any geometry! – at a distance R apart from each other. As R increases, the mutual inductance M will change as a function of R. Make an argument for the asymptotic behavior of M as a function of R at very large R, for example some function like $M(R) \sim e^{-R}$ (this is not the answer, but just demonstrating that we're looking for an asymptotic function).

   Solution: If the separation is very large compared to the characteristic size of each circuit, $R \gg s$, we can use the multipole expansion to write $\boldsymbol{B}$ as a power series in $(s/R)^k$. Recall that for the magnetic field, unlike the electric field, there is no monopole field where $\boldsymbol{E}_{\text{mono}} \sim 1/R^2$. Instead, the leading term in the magnetic field is the dipole term, $\boldsymbol{B}_{\text{dip}} \sim 1/R^3$ (unless the circuit is designed to make the dipole moment vanish).

   Therefore, at the location of circuit 2, we would compute (approximately)

   $$\Phi_2 = MI_1 = \int_{\text{circ. } 2} \boldsymbol{B}_1 \cdot d^2\boldsymbol{a}_2 \propto \frac{I_1}{R^3}\,. \tag{1}$$

   This suggests that $\boxed{M \propto R^{-3}}$ when $R \gg s$.

   Try to remember this magnetostatic result for when we get to radiation!

4. Much of the interstellar medium (ISM) is very low number density, typically $n \approx 1 \text{atom/cm}^3$ (you can assume it is entirely Hydrogen). There are magnetic fields permeating the galaxy with strengths like $B \approx 10\mu G$ (microGauss).

   (a) What is a typical energy density of the magnetic field in the ISM?
   Solution: Energy density in the magnetic field is $u_B = \frac{1}{2\mu_0}B^2 \approx 4 \times 10^{-13} J/m^3$ (in SI units) or $\approx 4 \times 10^{-12} erg/cm^3$ (in cgs units).

   (b) Suppose there is *equipartition* of energy between magnetic energy density and thermal energy density (which of course depends on density and temperature). What is a typical temperature of the ISM?

   Solution: Assuming equipartition, we would expect $u_B \approx u_{th}$, where the energy density in a thermal gas is $u_{th} = n\langle E\rangle$ where $\langle E\rangle$ is the average energy per particle and n is the number density. The average energy per degree of freedom is $\frac{1}{2}k_B T$ where k_B is Boltzmann's constant. Most of the gas in the ISM is a single Hydrogen atom, so it only has translation degrees of freedom (no rotational or internal), so $\langle E\rangle = \frac{3}{2}k_B T$. Therefore we expect $u_B = \frac{3}{2}nk_B T$ and can solve for temperature $T = \frac{B^2}{3\mu_0 n k_B} \approx 2 \times 10^4 K$. This is not too far off from temperatures of the "warm neutral medium."

5. **A highly conducting, magnetized plasma**. Consider a plasma with a conductivity[1] σ, charge density ρ (that varies throughout the plasma), and where at each point the particles are moving with velocity $\boldsymbol{v}$ (that also varies from place to place). Ohm's law says that the current density is

$$\boldsymbol{J} = \sigma \boldsymbol{f} = \sigma\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right). \tag{2}$$

   (a) Suppose the conductivity σ is taken to infinity, while the current density $\boldsymbol{J} = \rho\boldsymbol{v}$ remains finite. What relationship does this imply between the electromagnetic fields?
   Solution: For $\boldsymbol{J}$ to remain finite while $\sigma \to \infty$, we need $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \to 0$ at the same speed as $1/\sigma \to 0$. Thus we end up with

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}. \tag{3}$$

   (b) From the previous answer, what do you know about $\boldsymbol{E} \cdot \boldsymbol{B}$?
   Solution: Since $\boldsymbol{E}$ is the cross product of *something* with $\boldsymbol{B}$, the electric field must be perpendicular to the magnetic field, so

$$\boldsymbol{E} \cdot \boldsymbol{B} = 0. \tag{4}$$

   (c) Write $\boldsymbol{v}$ as the sum of two vectors, $\boldsymbol{v}_\parallel$ and $\boldsymbol{v}_\perp$ which are parallel and perpendicular to $\boldsymbol{B}$. Find an expression for $\boldsymbol{v}_\perp$ terms of $\boldsymbol{E}$ and $\boldsymbol{B}$.
   Solution: We write $\boldsymbol{v} = v_\parallel \hat{\boldsymbol{B}} + \boldsymbol{v}_\perp$ where $\boldsymbol{v}_\perp \cdot \boldsymbol{B} = 0$ and $\hat{\boldsymbol{B}} = \boldsymbol{B}/|B|$. Then in Eq. (3), only $\boldsymbol{v}_\perp$ contributes to the cross product, $\boldsymbol{E} = -\boldsymbol{v}_\perp \times \boldsymbol{B}$. This equation tells us that $\boldsymbol{v}_\perp$ makes a right angle with both $\boldsymbol{B}$ and $\boldsymbol{E}$, so it is proportional to $\boldsymbol{E} \times \boldsymbol{B}$ with some coefficient. You can plug in this ansatz and solve for the coefficient. Alternatively, you can take the cross product of the whole equation with $\boldsymbol{B}$,

$$\boldsymbol{B} \times \boldsymbol{E} = -\boldsymbol{B} \times (\boldsymbol{v}_\perp \times \boldsymbol{B}) \tag{5}$$

$$= -[\boldsymbol{v}_\perp(\boldsymbol{B} \cdot \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{v}_\perp \cdot \boldsymbol{B})] = -B^2 \boldsymbol{v}_\perp\,, \tag{6}$$

   since $\boldsymbol{v}_\perp$ is perpendicular to $\boldsymbol{B}$. Thus we have $\boldsymbol{v}_\perp = \boldsymbol{E} \times \boldsymbol{B}/B^2$.

   This scenario is actually applicable to many astrophysical plasmas! We will continue with this problem another week.

[1] We will not refer to the resistivity, $1/\sigma$, which is sometimes denoted ρ. Instead we reserve the symbol ρ for the charge density.

6. Griffiths 7.47 (3rd edition) [7.49 in the 4th edition] (getting $\boldsymbol{E}$ in terms of the vector potential).

Solution:

(a) (In the 3rd edition): In magnetostatic we have $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$. Biot-Savart says that a magnetic field that solves this is $\boldsymbol{B} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d^3r'$. In parallel, the equations for induced electric fields are $\nabla \cdot \boldsymbol{E} = 0$ (no charges, just induced fields) and $\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}$. So we can take Biot-Savart and replace $\boldsymbol{B}$ with $\boldsymbol{E}$ on the left-hand side if we replace $\mu_0 \boldsymbol{J}$ with $-\frac{\partial \boldsymbol{B}}{\partial t}$on the right-hand side. This gives the desired equation,

$$\boldsymbol{E} = \frac{-1}{4\pi} \frac{\partial}{\partial t} \int \frac{\boldsymbol{B}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d^3r' \,. \tag{7}$$

(b) From the earlier problem, $\boldsymbol{A}$ depends on $\boldsymbol{B}$ in the same way that $\boldsymbol{B}$ depends on $\mu_0 \boldsymbol{J}$. So replacing in Biot-Savart gives

$$\boldsymbol{A} = \frac{1}{4\pi} \int \frac{\boldsymbol{B}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d^3r' \,. \tag{8}$$

Comparing Eqs. (7) and (8) we see that $\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t$.

(c) Because electrodynamics is linear, we can superimpose the following two results. First, just due to stationary charge density, which is not changing, we get an electric field that vanishes inside the shell, and outside is given by $\boldsymbol{E}_{Coul.} = \hat{\boldsymbol{r}} \frac{Q}{4\pi\epsilon_0 r^2}$, where $Q = 4\pi R^2$, or $\boldsymbol{E}_{Coul.} = \hat{\boldsymbol{r}} \frac{\sigma R^2}{\epsilon_0 r^2}$. Second, from the currents which slowly change with time. We can take the result for $\boldsymbol{A}_{Farad.}$ from Ex. 5.11, as long as the frequency $\omega(t)$ changes very slowly in time (i.e. $d\omega/dt << \omega^2$). That result was

$$\boldsymbol{A}_{Farad.} = \begin{cases} \frac{\mu_0 R \omega \sigma}{3} r \sin\theta \hat{\boldsymbol{\phi}} \,, & (r < R), \\ \frac{\mu_0 R^4 \omega \sigma}{3} \frac{\sin\theta}{r^2} \hat{\boldsymbol{\phi}} \,, & (r > R). \end{cases} \tag{9}$$

Now we consider $\omega = \omega(t)$ to be a function of time, and use $\boldsymbol{E}_{Farad.} = -\partial \boldsymbol{A}_{Farad.}/\partial t$ to get the contribution from the current. Finally, the total electric field is the sum $\boldsymbol{E} = \boldsymbol{E}_{Coul.} + \boldsymbol{E}_{Farad.}$,

$$\boldsymbol{E} = \begin{cases} \frac{\mu_0 R \dot{\omega} \sigma}{3} r \sin\theta \hat{\boldsymbol{\phi}} \,, & (r < R), \\ \frac{\sigma R^2}{\epsilon_0 r^2} \hat{\boldsymbol{r}} + \frac{\mu_0 R^4 \dot{\omega} \sigma}{3} \frac{\sin\theta}{r^2} \hat{\boldsymbol{\phi}} \,, & (r > R). \end{cases} \tag{10}$$