UNIVERSITY OF MISSISSIPPI

Department of Physics and Astronomy

Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2019

Problem Set 10 — SOLUTIONS

Due: Friday, May 10, 2019, by 5PM

Material: The final covers all the material so far.

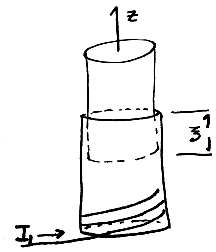
Due date: Friday, May 10, 2019 by 5PM to 205 Lewis Hall. If my door is closed, please slide the exam under my door. Late exams will require extenuating circumstances.

Logistics: The exam consists of this page plus 2 pages of questions. Do not look at the problems until you are ready to start it.

Time: The work might expand to eat up as much time as you allot – therefore I highly recommend you restrict yourself to no more than 5 hours cumulative time spent on these problems. You may take as many breaks as you like, not counted against the 5 hours. **You should not be consulting references, working on the problems, or discussing with others during the breaks**.

Resources: The final is **not collaborative**. All questions must be done on your own, without consulting anyone else. You may consult your own notes (both in-class and notes on this class you or a colleague in the class have made), the textbook by Griffiths, and solution sets on the course website. **You may not consult any other material**, including other textbooks, the web (except for the current Phys. 402 website), material from previous years' Phys. 402 or any other classes, or copies you have made of such material, or any other resources. Calculators and symbolic manipulation programs are not allowed.

1. **Tunable inductor**. For a traditional analog radio, you need to be able to tune a resonant circuit. One way to accomplish this is with an LC resonator, where the inductance of an inductor can be varied by inserting/removing a core (by turning a screw).
Suppose we have a solenoid of length ℓ, with N total turns, giving $n \equiv N/\ell$ turns per unit length. Along its axis is a core of some linear material (say it's Aluminum) with permeability μ (where $\mu > \mu_0$), and it can be inserted a distance ξ into the solenoid, ranging over $0 \leq \xi \leq \ell$.



   (a) Compute the self-inductance L of this solenoid, as a function of the insertion depth ξ.
   Solution: The magnetic field in the air section of the solenoid is $B_{z,air} = \mu_0 n I$, while the magnetic field in the Aluminum section is $B_{z,Al} = \mu n I$. The flux through an individual loop in either section is is $\Phi_1 = B_z \pi r^2$, where B_z is either the value in air or Aluminum, respectively. The flux total flux is $\Phi = \Phi_{air} + \Phi_{Al} = N_{air}\Phi_{1,air} + N_{Al}\Phi_{1,Al}$, where there are $N_{air} = (\ell - \xi)n$ loops in the air section, and $N_{Al} = \xi n$ loops in the Aluminum section. Therefore the total flux is $\Phi = \pi r^2 n^2 I[(\ell - \xi)\mu_0 + \xi\mu]$. The self-inductance L is the constant of proportionality in $\Phi = LI$, so we can identify $L = \pi r^2 n^2 [(\ell - \xi)\mu_0 + \xi\mu]$.

   (b) In terms of the magnetic susceptibility χ_m, what is the ratio of max and min inductances, $L_{\max}/L_{\min}$, of this tunable inductor?
   Solution: Writing $\mu = \mu_0(1 + \chi_m)$, the inductance is $L = \pi r^2 n^2 \mu_0[(\ell - \xi) + \xi(1 + \chi_m)] = \pi r^2 n^2 \mu_0[\ell + \xi\chi_m]$. The minimum will be $L_{\min} = L(\chi = 0) = \pi r^2 n^2 \mu_0 \ell$, and the maximum will be $L_{\max} = L(\chi = \ell) = \pi r^2 n^2 \mu_0 \ell(1 + \chi_m)$. Therefore their ratio is simply $L_{\max}/L_{\min} = (1 + \chi_m)$.

   (c) Find the energy $E_B(\xi)$ stored in the magnetic field inside the solenoid, as a function of ξ.
   Solution: The energy is $E_B = \frac{1}{2\mu_0}\int B^2 d^3r$. The magnetic field is uniform in each region, so the integral is just a product, $E_B = \frac{1}{2\mu_0}[B_{air}^2 V_{air} + B_{Al}^2 V_{al}] = \frac{\pi r^2 \mu_0 n^2 I^2}{2}[(\ell - \xi) + \xi(1 + \chi_m)^2]$.

2. **Gradually changing waveguide**. Suppose we have a hollow waveguide running in the z direction for several kilometers. At every z the cross-section is a rectangle with sides a, b where $a \geq b$. Suppose we send in a wave that excites the TE_{mn} mode with some frequency ω and wavenumber k that satisfy the dispersion relation [Griffiths Eq. (9.187)]

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]} = \frac{1}{c^2}\sqrt{\omega^2 - \omega_{mn}^2}\,, \tag{1}$$

and the amplitude is given by some B_0 in [Griffiths Eq. (9.186)]

$$B_z = B_0 \cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})\,. \tag{2}$$

   (a) What is the average energy flux $\langle \boldsymbol{S} \rangle$ (averaged over one period of the wave)? What is the averaged energy flux through the whole cross-section, $\int \langle \boldsymbol{S} \rangle \cdot d\boldsymbol{a}$? (The result of Griffiths' problem 9.11 [time-averaging a product using complex exponentials] might be helpful).
   Solution: We want to find $\langle \boldsymbol{S} \rangle = \frac{1}{2\mu_0}\tilde{\boldsymbol{E}} \times \tilde{\boldsymbol{B}}*$. The complex fields are $\tilde{\boldsymbol{E}} = \tilde{\boldsymbol{E}}_0 e^{i(kz-\omega t)}$ and $\tilde{\boldsymbol{B}}^* = \tilde{\boldsymbol{B}}_0^* e^{-i(kz-\omega t)}$. For the TE_{mn} mode, $E_z = 0$, and the entire solution is determined in terms of B_z. We find $E_{x,y}$ and $B_{x,y}$ from various derivatives of B_z [Eq. (9.180)], finding

$$B_x^* = \frac{-ik}{(\omega/c)^2 - k^2}(\frac{-m\pi}{a})B_0 \sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b}) \tag{3}$$

$$B_y^* = \frac{-ik}{(\omega/c)^2 - k^2}(\frac{-n\pi}{b})B_0 \cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b}) \tag{4}$$

$$E_x = \frac{-i\omega}{(\omega/c)^2 - k^2}(\frac{-n\pi}{b})B_0 \cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b}) \tag{5}$$

$$E_y = \frac{-i\omega}{(\omega/c)^2 - k^2}(\frac{-m\pi}{a})B_0 \sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b})\,. \tag{6}$$

Now computing the cross product and averaging, we get

$$\langle \boldsymbol{S} \rangle = \frac{1}{2\mu_0} \frac{B_0^2}{(\omega/c)^2 - k^2} \Bigg\{ \frac{i\pi\omega m}{a} \sin(\frac{m\pi x}{a}) \cos(\frac{m\pi x}{a}) \cos^2(\frac{n\pi y}{b}) \hat{\boldsymbol{x}} \tag{7}$$
$$+ \frac{i\pi\omega n}{b} \cos^2(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \cos(\frac{n\pi y}{b}) \hat{\boldsymbol{y}}$$
$$+ \frac{\omega k \pi^2}{(\omega/c)^2 - k^2} \left[(\frac{n}{b})^2 \cos^2(\frac{m\pi x}{a}) \sin^2(\frac{n\pi y}{b}) + (\frac{m}{a})^2 \sin^2(\frac{m\pi x}{a}) \cos^2(\frac{n\pi y}{b}) \right] \hat{\boldsymbol{z}} \Bigg\} .$$

Finally we integrate over x from 0 to a, and over y from 0 to b, to find

$$\int \langle \boldsymbol{S} \rangle \cdot d\boldsymbol{a} = \frac{1}{8\mu_0} \frac{\omega k \pi^2 B_0^2}{[(\omega/c)^2 - k^2]^2} ab \left[(\frac{m}{a})^2 + (\frac{n}{b})^2 \right] = \frac{\omega k a b c^2}{8\mu_0 \omega_{mn}^2} B_0^2 . \tag{8}$$

Now suppose that this waveguide's cross-section changes very slowly in z, so that $a = a(z)$ and $b = b(z)$. Very slowly here means that $\frac{1}{a}\frac{da}{dz} \ll k$ and similarly for b. For simplicity we will assume that the aspect ratio a/b remains constant.

(b) Will ω change with z? What about k?

Solution: The frequency ω will not change, but k will.

(c) Now the amplitude $B_0(z)$ will have to slowly vary with z. Find a differential equation that would allow you to solve for $B_0(z)$ if somebody gave you $a(z)$ (and thus they are also giving you $b(z)$ since their ratio is constant).

Solution: The energy flux through each z must be the same for energy to be conserved. Therefore we must have

$$\frac{d}{dz} \int \langle \boldsymbol{S} \rangle \cdot d\boldsymbol{a} = 0 . \tag{9}$$

Therefore we have

$$0 = \frac{d}{dz} \left[\frac{\omega k a b c^2}{8\mu_0 \omega_{mn}^2} B_0^2 \right] . \tag{10}$$

Inside the brackets, a, b, k, ω_{mn}, and B_0 are all functions of z.

(d) Find a simple combination of a, B_0, and k that is constant along z.

Solution: Start from Eq. (10). Then replace b with $b = \epsilon a$ where $\epsilon = b/a$ is a constant. We can ignore several constants that can be taken out of the derivative and divided out. Now we have

$$0 = \frac{d}{dz} \left[\frac{k a^2}{(m/a)^2 + (n/\epsilon a)^2} B_0^2 \right] \tag{11}$$
$$0 = \frac{d}{dz} \left[\frac{k a^4}{m^2 + (n/\epsilon)^2} B_0^2 \right] . \tag{12}$$

The denominator is a constant, so we have found $ka^4 B_0^2$ is a constant along z.

(e) What will happen if a gradually shrinks too small?

Solution: If a becomes too small, ω_{mn} will grow to be larger than ω, and the mode will not be able to propagate any more. It will reflect back down the waveguide in the opposite direction. Notice that as ω_{mn} approaches ω, $k \to 0$, so it is impossible shrink a to this point adiabatically (satisfying $\frac{1}{a}\frac{da}{dz} \ll k$).

3. **Integral identities**. For the following problems, you can assume that as you go to very large distances, the electric field decays as $1/r^2$, and the magnetic field decays as $1/r^3$.

   (a) How quickly can the vector potential $\boldsymbol{A}$ decay at large r?
   Solution: It can decay as $1/r^2$; one derivative of $\boldsymbol{A}$ gives $\boldsymbol{B}$.

   (b) Prove the following integral identity, for any two vector fields $\boldsymbol{V}, \boldsymbol{W}$ integrated over volume $\mathcal{V}$:

$$\int_{\mathcal{V}} \boldsymbol{W} \cdot (\nabla \times \boldsymbol{V})\, d^3r = \int_{\partial\mathcal{V}} (\boldsymbol{V} \times \boldsymbol{W}) \cdot d\boldsymbol{a} + \int_{\mathcal{V}} \boldsymbol{V} \cdot (\nabla \times \boldsymbol{W})\, d^3r\,. \tag{13}$$

   Solution: This comes simply from rearranging the product rule

$$\nabla \cdot (\boldsymbol{V} \times \boldsymbol{W}) = \boldsymbol{W} \cdot (\nabla \times \boldsymbol{V}) - \boldsymbol{V} \cdot (\nabla \times \boldsymbol{W})\,, \tag{14}$$

   then integrating over $\mathcal{V}$, and applying the divergence theorem on the appropriate term.

   (c) Now combine everything to show that the following integral over all space and time vanishes:

$$\int_{-\infty}^{+\infty} \int_{\text{All space}} (\boldsymbol{E} \cdot \boldsymbol{B})\, d^3r\, dt = 0\,. \tag{15}$$

   [Hint: use the potential formulation, and assume that the fields also vanish as $t \to \pm\infty$.]
   Solution: We have $\boldsymbol{E} = -\nabla V - \frac{\partial}{\partial t}\boldsymbol{A}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Our integral is now

$$\int_{-\infty}^{+\infty} \int_{\text{All space}} [-\nabla V \cdot (\nabla \times \boldsymbol{A}) - \frac{\partial \boldsymbol{A}}{\partial t} \cdot (\nabla \times \boldsymbol{A})]\, d^3r\, dt\,. \tag{16}$$

   In the first term, integrated by parts to move the gradient from V onto $\boldsymbol{B}$. The new integrand vanished because $\boldsymbol{B}$ is divergence-free. The boundary term vanishes because the fields decay sufficiently rapidly.
   Now we have to handle

$$\int_{-\infty}^{+\infty} \int_{\text{All space}} [-\frac{\partial \boldsymbol{A}}{\partial t} \cdot (\nabla \times \boldsymbol{A})]\, d^3r\, dt\,. \tag{17}$$

   Using the identity in Eq. (13), we can rewrite this as

$$\int_{-\infty}^{+\infty} \int_{\text{All space}} [-\frac{\partial \boldsymbol{A}}{\partial t} \cdot (\nabla \times \boldsymbol{A})]\, d^3r\, dt = \int_{-\infty}^{+\infty} \int_{\text{Boundary}} [-\frac{\partial \boldsymbol{A}}{\partial t} \times \boldsymbol{A}] \cdot d\boldsymbol{a}\, dt + \int_{-\infty}^{+\infty} \int_{\text{All space}} [-\boldsymbol{A} \cdot (\nabla \times \frac{\partial \boldsymbol{A}}{\partial t})]\, d^3r\, dt$$

   In the second integral on the RHS, the time derivative commutes with the curl. Then integrate by parts the time derivative to put it on the first factor. Now this term is the same as the one on the LHS, except for a sign; they combine. Now we have

$$\int_{-\infty}^{+\infty} \int_{\text{All space}} [\frac{\partial \boldsymbol{A}}{\partial t} \cdot (\nabla \times \boldsymbol{A})]\, d^3r\, dt = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{\text{Boundary}} [\frac{\partial \boldsymbol{A}}{\partial t} \times \boldsymbol{A}] \cdot d\boldsymbol{a}\, dt \tag{18}$$

   and we want to show that this vanishes. But the term on the right hand side is integrated over an arbitrarily large 2-sphere with radius R. Each factor of $\boldsymbol{A}$ inside decays as $1/R^2$, and the area grows as R^2, so the entire integral decays as $1/R^2$ overall. Thus it vanishes as $R \to \infty$.

4. **Radiation reaction in an earlier problem**. In an earlier problem, we considered a rotor of length $2b$ laying in the $x - y$ plane, with a charge $+q$ at one end and $-q$ at the other end, spinning about the z axis at angular frequency ω. In that problem, we found the dipole radiation carried an energy flux (integrated over all angles):

$$\langle \frac{dE}{dt} \rangle = \frac{2\mu_0 q^2 b^2 \omega^4}{3\pi c} \,. \tag{19}$$

Let us now give each of these charges a mass m, and let the rotor not contribute to the moment of inertia.

(a) Since energy is leaving, the spin rate $\omega(t)$ will slowly decrease. Find a differential equation for $d\omega/dt$ and solve for $\omega(t)$.

Solution: Energy conservation. All the energy is in rotational kinetic energy, $E = \frac{1}{2} I\omega^2$, with $I = 2mb^2$. Combining we get

$$\frac{1}{2}(2mb^2)2\omega \frac{d\omega}{dt} = -\frac{2\mu_0 q^2 b^2 \omega^4}{3\pi c} \,. \tag{20}$$

Solving,

$$\frac{d\omega}{dt} = -\frac{\mu_0 q^2 \omega^3}{3\pi mc} \equiv -\alpha\omega^3 \,, \tag{21}$$

where $\alpha \equiv \mu_0 q^2/3\pi mc$. This can be separated to integrate,

$$\frac{1}{2\omega^2} = \alpha t + \frac{1}{2\omega_0} \tag{22}$$

where ω_0 is the value of the spin at time 0. The solution is

$$\omega(t) = \frac{\omega_0}{\sqrt{1 + 2\alpha t \omega_0^2}} \,. \tag{23}$$

(b) Now compute the radiation-reaction force on each particle. Use this to compute the torque on the system and so find another expression for $d\omega/dt$. Do they give the same result? Why or why not?

Solution: The Abraham-Dirac-Lorentz formula is

$$\boldsymbol{F}_{RR} = \frac{\mu_0 q^2}{6\pi c} \dot{\boldsymbol{a}} \,. \tag{24}$$

We have $\boldsymbol{a}$ pointing radially inward with magnitude $\omega^2 b$, so $\dot{\boldsymbol{a}}$ points in the opposite direction of velocity, with magnitude $\omega^3 b$. Each charged particle contributes to the torque the same amount, giving

$$\tau_{RR} = -2\frac{\mu_0 q^2}{6\pi c}\omega^3 b^2 \,. \tag{25}$$

Torque is the time derivative of angular momentum, so we have

$$-\frac{\mu_0 q^2}{3\pi c}\omega^3 b^2 = \frac{d}{dt} I\omega = (2mb^2)\frac{d\omega}{dt} \,, \tag{26}$$

$$\implies \quad \frac{d\omega}{dt} = -\frac{\mu_0 q^2}{6\pi mc}\omega^3 \,. \tag{27}$$

Notice that this is off by a factor of 2 from the result we got in part (a). The calculation in this part is *only* from the radiation-reaction force, and did not include the Lorentz force on particle 1 due to the field of particle 2, and vice versa. That additional force accounts for the factor of 2. See the related discussion at the end of Griffiths' section 11.2.3.

5. Griffiths problem 12.47 (transform a plane electromagnetic wave to a new frame).

Solution:

(a) The electric and magnetic fields are

$$\boldsymbol{E} = E_0 \cos(kx - \omega t)\hat{\boldsymbol{y}}\,, \tag{28}$$

$$\boldsymbol{B} = \frac{E_0}{c} \cos(kx - \omega t)\hat{\boldsymbol{z}}\,, \tag{29}$$

where $k = \omega/c$.

(b) We use the general transformation of $\boldsymbol{E}, \boldsymbol{B}$ fields when one boosts in the x direction; but the only non-vanishing components we have to start with are E_y and B_z. The result is

$$\bar{E}_x = 0\,, \qquad \bar{E}_y = \gamma(E_y - vB_z)\,, \qquad \bar{E}_z = 0\,, \tag{30}$$

$$\bar{B}_x = 0\,, \qquad \bar{B}_y = 0\,, \qquad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y)\,. \tag{31}$$

Plugging in the fields from Eq. (28) and (29),

$$\bar{E}_y = \alpha E_0 \cos(kx - \omega t)\,, \qquad \bar{B}_z = \alpha \frac{E_0}{c} \cos(kx - \omega t)\,, \tag{32}$$

where

$$\alpha \equiv \gamma\left(1 - \frac{v}{c}\right) = \sqrt{\frac{1 - v/c}{1 + v/c}}\,. \tag{33}$$

Now we need to go to the barred spacetime coordinates via the Lorentz transformation, $x = \gamma(\bar{x} + v\bar{t})$, $t = \gamma(\bar{t} + \frac{v}{c^2}\bar{x})$, so

$$kx - \omega t = \bar{k}\bar{x} - \bar{\omega}\bar{t} \tag{34}$$

where

$$\bar{k} \equiv \gamma(k - \frac{\omega v}{c^2}) = \alpha k\,, \qquad \bar{\omega} \equiv \gamma\omega(1 - v/c) = \alpha\omega\,. \tag{35}$$

So in summary, we have

$$\bar{\boldsymbol{E}} = \bar{E}_0 \cos(\bar{k}\bar{x} - \bar{\omega}\bar{t})\hat{\boldsymbol{y}}\,, \tag{36}$$

$$\bar{\boldsymbol{B}} = \frac{\bar{E}_0}{c} \cos(\bar{k}\bar{x} - \bar{\omega}\bar{t})\hat{\boldsymbol{z}}\,, \tag{37}$$

where the barred quantities are $\bar{E}_0 = \alpha E_0$, $\bar{k} = \alpha k$, $\bar{\omega} = \alpha\omega$, where $\alpha = \sqrt{\frac{1-v/c}{1+v/c}}$.

(c) The new frequency is $\bar{\omega} = \alpha\omega = \sqrt{\frac{1-v/c}{1+v/c}}\omega$. This is the *Doppler shift*. The wavelength is $\bar{\lambda} = 2\pi/\bar{k} = 2\pi/\alpha k = \lambda/\alpha$. The phase velocity in this frame is $\bar{v} = \bar{\omega}/\bar{k} = c$. As required, the speed of light is the same in all inertial frames – that was the assumption for deriving the Lorentz transformations.

(d) Intensity is proportional to E^2, so the ratio of intensities is

$$\frac{\bar{I}}{I} = \frac{\bar{E}_0^2}{E_0^2} = \alpha^2 = \frac{1 - v/c}{1 + v/c}\,. \tag{38}$$

As you approach the speed of light, the amplitude, frequency, and intensity of light all go to zero, so it will be more and more difficult to see the light.