UNIVERSITY OF MISSISSIPPI
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Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2019

Problem Set 8 — SOLUTIONS

Due: Friday, Apr. 26, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. **Radiation from charges in circular motion**. Suppose we have a rotor of length $2b$ that lies in the $x-y$ plane, centered on the $\hat{z}$ axis. We attach a charge $+q$ to one end, and an opposite charge $-q$ to the opposite end, and then spin this rotor around the $\hat{z}$ axis (clockwise as seen from above) at an angular frequency of ω radians per second. At time $t=0$ it lies along the x axis.

   (a) Write the trajectories $\boldsymbol{\xi}_+(t)$ of the positive charge and $\boldsymbol{\xi}_-(t)$ of the negative charge. What is the charge distribution $\rho(t,\boldsymbol{r})$? (There will be delta functions!)

   Solution: A compact way to encode the circular motion of both particles is to use the real parts of complex trajectories,

$$\boldsymbol{\xi}_+(t) = \mathrm{Re}[be^{i\omega t}(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})] \tag{1}$$
$$\boldsymbol{\xi}_-(t) = -\boldsymbol{\xi}_+(t)\,. \tag{2}$$

   The charge distribution is given by a delta function at the location of each particle,

$$\rho(t,\boldsymbol{r}) = +q\delta^{(3)}(\boldsymbol{r}-\boldsymbol{\xi}_+(t)) - q\delta^{(3)}(\boldsymbol{r}-\boldsymbol{\xi}_-(t))\,. \tag{3}$$

   (b) Compute the first three charge *moments* of this distribution, as a function of time: the 0th moment (charge), the 1st moment (charge dipole), and the 2nd moment (charge quadrupole). The charge quadrupole moment $\overleftrightarrow{\boldsymbol{\mathcal{Q}}}$ is a symmetric tensor defined similarly to the charge monopole Q and dipole moment $\boldsymbol{p}$,

$$Q(t) = \int \rho(t,\boldsymbol{r}')d^3\boldsymbol{r}' \tag{4}$$
$$p_i(t) = \int \rho(t,\boldsymbol{r}')r'_i d^3\boldsymbol{r}' \tag{5}$$
$$\mathcal{Q}_{ij}(t) = \int \rho(t,\boldsymbol{r}')r'_i r'_j d^3\boldsymbol{r}'\,. \tag{6}$$

   [**Note** 1: The definition of quadrupole moment I wrote here differs from what Griffiths writes in Chapter 3. The convention I'm using is more in line with what's in the general relativity literature. Note 2: If you find this problem confusing, you probably want to review chapter 3, and perhaps try problem 3.45.]

   Solution: The total charge is very easy, $Q(t) = \int \rho(t,\boldsymbol{r}')d^4\boldsymbol{r}' = 0$. The other two are slightly harder.

$$\boldsymbol{p}(t) = \int \rho(t,\boldsymbol{r}')\boldsymbol{r}'d^3\boldsymbol{r}' = q\boldsymbol{\xi}_+(t) - q\boldsymbol{\xi}_-(t) = 2bq\mathrm{Re}[e^{i\omega t}(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})]\,, \tag{7}$$
$$\mathcal{Q}^{ij}(t) = \int \rho(t,\boldsymbol{r}')r'^i r'^j d^3\boldsymbol{r}' = q\xi_+^i(t)\xi_+^j(t) - q\xi_-^i(t)\xi_-^j(t)\,. \tag{8}$$

(c) How would you define the octupole moment of a charge distribution?

Solution: Following the progression, it is natural to define the octupole moment as

$$\mathcal{O}_{ijk}(t) = \int \rho(t, \boldsymbol{r'}) r'_i r'_j r'_k d^3\boldsymbol{r'} \,. \tag{9}$$

(d) Find the second time derivative of the dipole moment, $\ddot{\boldsymbol{p}}$.

Solution: We only need to take the time derivative of $e^{i\omega t}$, so we get

$$\ddot{\boldsymbol{p}} = -2bq\omega^2 \text{Re}[e^{i\omega t}(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})] = -\omega^2 \boldsymbol{p} \,. \tag{10}$$

(e) Using Griffiths' result for the dipole radiation from a general charge distribution, find the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{B}$ produced by this system, at a large distance $r \gg \lambda$. [Note that Griffiths' equation in terms of $\ddot{\boldsymbol{p}}$ is correct for any orientation of coordinate system. However his later equations in terms of $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ have aligned the z axis along the (retarded) direction of $\ddot{\boldsymbol{p}}$, so if you wanted to apply those, you'd need to constantly change your basis!]

Solution: Dipole radiation from an arbitrary source produces radiative $\boldsymbol{E}, \boldsymbol{B}$ fields given by

$$\boldsymbol{E} = \frac{\mu_0}{4\pi r} \hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}}) \,, \tag{11}$$

$$\boldsymbol{B} = \frac{-\mu_0}{4\pi r c} \hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}} \,, \tag{12}$$

where $\boldsymbol{p}$ is evaluated at the retarded time $t_0 \equiv t - r/c$. Let's compute those cross products, which is easiest by converting $\boldsymbol{p}$ into the basis of $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ at the far-zone field point with coordinates (r, θ, ϕ). Using the transformations in Griffiths we can write

$$\boldsymbol{p}(t) = 2bq\text{Re}[e^{i\omega t}(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})] = 2bq\text{Re}[e^{i\omega t + i\phi}\{\hat{\boldsymbol{r}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta + i\hat{\boldsymbol{\phi}}\}] \,. \tag{13}$$

Now we can find the cross products as

$$\hat{\boldsymbol{r}} \times \boldsymbol{p} = 2bq\text{Re}[e^{i\omega t + i\phi}\{\hat{\boldsymbol{\phi}} \cos\theta - i\hat{\boldsymbol{\theta}}\}] \,, \tag{14}$$

$$\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \boldsymbol{p}) = 2bq\text{Re}[e^{i\omega t + i\phi}\{-\hat{\boldsymbol{\theta}} \cos\theta - i\hat{\boldsymbol{\phi}}\}] \,. \tag{15}$$

Thus we have the dipole radiation as

$$\boldsymbol{E} = \frac{-2bq\omega^2\mu_0}{4\pi r} \text{Re}[e^{i\omega t + i\phi}\{-\hat{\boldsymbol{\theta}} \cos\theta - i\hat{\boldsymbol{\phi}}\}] \tag{16}$$

$$\boldsymbol{E} = \frac{-2bq\omega^2\mu_0}{4\pi r} \left[-\hat{\boldsymbol{\theta}} \cos\theta \cos(\omega t + \phi) - \hat{\boldsymbol{\phi}} \cos(\omega t + \phi + \pi/2)\right] \,, \tag{17}$$

$$\boldsymbol{B} = \frac{+2bq\omega^2\mu_0}{4\pi r c} \text{Re}[e^{i\omega t + i\phi}\{\hat{\boldsymbol{\phi}} \cos\theta - i\hat{\boldsymbol{\theta}}\}] \tag{18}$$

$$\boldsymbol{B} = \frac{+2bq\omega^2\mu_0}{4\pi r c} \left[+\hat{\boldsymbol{\phi}} \cos\theta \cos(\omega t + \phi) - \hat{\boldsymbol{\theta}} \cos(\omega t + \phi + \pi/2)\right] \,. \tag{19}$$

(f) What is the polarization of the electromagnetic radiation along the positive z axis? Negative z axis? What about in the $x - y$ plane?

Solution: The $x - y$ plane is at $\theta = \pi/2$, where $\cos\theta = 0$. There, we see that the $\boldsymbol{E}$ field is purely in the $\hat{\boldsymbol{\phi}}$ direction, and the $\boldsymbol{B}$ field is purely in the $\hat{\boldsymbol{\theta}}$ direction. Thus the radiation is linearly polarized in the $x - y$ plane.

Meanwhile, on the z axis is either at $\theta = 0$ (the North pole) or $\theta = \pi$ (the South pole). At both places, the $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ basis vectors are not continuous. Therefore let's go back to the $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ basis.

On the North pole $(\theta = 0)$, $\hat{\boldsymbol{r}} = \hat{\boldsymbol{z}}$, so we need

$$\hat{\boldsymbol{z}} \times \boldsymbol{p} = 2bq\text{Re}[e^{i\omega t}(\hat{\boldsymbol{y}} - i\hat{\boldsymbol{x}})] \,, \tag{20}$$

$$\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \boldsymbol{p}) = 2bq\text{Re}[e^{i\omega t}(-\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})] \,. \tag{21}$$

Therefore our solution will look like

$$\boldsymbol{E} = \frac{-2bq\omega^2\mu_0}{4\pi r}\mathrm{Re}[e^{i\omega t}(-\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})] \tag{22}$$

$$\boldsymbol{B} = \frac{+2bq\omega^2\mu_0}{4\pi rc}\mathrm{Re}[e^{i\omega t}(\hat{\boldsymbol{y}} - i\hat{\boldsymbol{x}})]\,. \tag{23}$$

This is circularly polarized, and both the electric and magnetic field are rotating clockwise about the $\hat{\boldsymbol{z}}$ axis as viewed from above, hence left circularly polarized. This makes sense, since an observer at the North pole would observe the particles rotating clockwise.

Meanwhile on the South pole ($\theta = \pi$), $\hat{\boldsymbol{r}} = -\hat{\boldsymbol{z}}$. We can do the cross products again, or just note that we can reuse our previous calculations and flip the sign of $\boldsymbol{B}$ (and flip the sign of $\boldsymbol{E}$ twice, which is no sign flip at all). Therefore we get

$$\boldsymbol{E} = \frac{-2bq\omega^2\mu_0}{4\pi r}\mathrm{Re}[e^{i\omega t}(-\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})] \tag{24}$$

$$\boldsymbol{B} = \frac{+2bq\omega^2\mu_0}{4\pi rc}\mathrm{Re}[e^{i\omega t}(-\hat{\boldsymbol{y}} + i\hat{\boldsymbol{x}})]\,. \tag{25}$$

This is also rotating about $\hat{\boldsymbol{z}}$ clockwise as viewed from the positive z direction, but the wave is going in the negative z direction. Viewed from below it is counterclockwise, hence right circularly polarized.

(g) Find the Poynting vector $\boldsymbol{S}$, and its period average $\langle \boldsymbol{S} \rangle$ at this large distance. Compare the angular dependence of the radiation pattern in this system with the one we saw before (a center-fed split dipole antenna).

Solution: Let's take the real fields, Eqs. (17) and (18), to plug into

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B}\,. \tag{26}$$

Plugging in we get

$$\boldsymbol{S} = \frac{1}{\mu_0}\left(\frac{+2bq\omega^2\mu_0}{4\pi rc}\right)^2 \left[\cos^2(\theta)\cos^2(\omega t + \phi) + \cos^2(\omega t + \phi + \pi/2)\right]\hat{\boldsymbol{r}}\,. \tag{27}$$

Taking the average over one period $T = 2\pi/\omega$, we get

$$\langle \boldsymbol{S} \rangle = \frac{b^2 q^2 \omega^4 \mu_0}{8\pi r^2}(1 + \cos^2\theta)\,. \tag{28}$$

(h) What is the total power emitted in radiation?

Solution: We have to integrate $\langle \boldsymbol{S} \rangle$ across the whole sphere, to get

$$P = \frac{dE}{dt} = \int \langle \boldsymbol{S} \rangle \cdot d\boldsymbol{a} = \int S_r r^2 d\cos\theta d\phi \tag{29}$$

$$= \frac{b^2 q^2 \omega^4 \mu_0}{8\pi} 2\pi \int_{-1}^{+1} (1 + \cos^2\theta) d\cos\theta \tag{30}$$

$$P = \frac{2b^2 q^2 \omega^4 \mu_0}{3}\,. \tag{31}$$

(i) What would happen if you replaced the negative charge with another positive charge?

Solution: The dipole moment would vanish, so there would also be no dipole radiation. Instead the radiation field would be dominated by octupole radiation, determined from the third time derivative of the octupole moment.

2. **Getting higher multipoles**. In this problem we will extend the derivation from dipole to higher multipole order. We will do so for the scalar wave equation,

$$\Box f(t, \boldsymbol{r}) = S(t, \boldsymbol{r})\,, \tag{32}$$

which has the solution

$$f(t, \boldsymbol{r}) = \frac{-1}{4\pi} \int \frac{S(t_r, \boldsymbol{r}')}{\mathscr{r}} d^3\boldsymbol{r}'\,, \tag{33}$$

where as before $t_r = t - \frac{1}{c}\mathscr{r}$.

(a) Show how to perform a Taylor series expansion on the first argument of S, so that instead of evaluating S at different retarded times, we evaluate various time derivatives of S all at the single time $t_0 \equiv t - \frac{r}{c}$.

Solution: For now we will suppress the second argument $\boldsymbol{r}'$, and write $S(t_r) = S(t_0 + (t_r - t_0))$, where $(t_r - t_0)$ is supposed to be small. Then we can expand

$$S(t_r) = \sum_{n=0}^{\infty} \frac{(t_r - t_0)^n}{n!} \frac{\partial^n S}{\partial t^n}\bigg|_{t=t_0}\,. \tag{34}$$

Now if we plug in the definitions of t_r and t_0, notice that the difference is $t_r - t_0 = t - \frac{\mathscr{r}}{c} - (t - \frac{r}{c}) = \frac{r - \mathscr{r}}{c}$. Therefore we can express the Taylor series as

$$S(t_r) = \sum_{n=0}^{\infty} \frac{(r - \mathscr{r})^n}{n! c^n} \frac{\partial^n S}{\partial t^n}\bigg|_{t=t_0}\,. \tag{35}$$

(b) In the above Taylor series, you should see the combination $(r - \mathscr{r})$ appearing. Using the series expansion $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(x^3)$, find the first two nonvanishing terms of $(r - \mathscr{r})$. This will be in terms of r, r', and $\cos\theta' = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}'$.

Solution: From the definition of $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$, we know that the magnitude can be written as

$$\mathscr{r} = \sqrt{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}}} = \sqrt{r^2 + (r')^2 - 2\boldsymbol{r} \cdot \boldsymbol{r}'} = \sqrt{r^2 + (r')^2 - 2rr'\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}'}\,. \tag{36}$$

Since $r \gg r'$, we want to use the ratio r'/r as a small parameter for expansions. Therefore factor out an r from inside the square root,

$$\mathscr{r} = r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}'} \tag{37}$$

$$\mathscr{r} \approx r\left[1 + \frac{1}{2}\left(-2\frac{r'}{r}\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}'\right) + \mathcal{O}\left(\frac{r'}{r}\right)^2\right]\,. \tag{38}$$

Finally we get the difference

$$r - \mathscr{r} \approx r'\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}' = \hat{\boldsymbol{r}} \cdot \boldsymbol{r}'\,. \tag{39}$$

(c) Combine your results to prove that the $\frac{1}{r}$ part of the solution for f is given by

$$f(t, \boldsymbol{r}) = \frac{-1}{4\pi}\frac{1}{r}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^n \int S(t_0, \boldsymbol{r}')(\hat{\boldsymbol{r}} \cdot \boldsymbol{r}')^n\, d^3\boldsymbol{r}' + \mathcal{O}(r^{-2})\,. \tag{40}$$

The nth term in the sum is determined by n derivatives of the nth multipole moment of the source.

Solution: So far we have

$$f(t, \boldsymbol{r}) \approx \frac{-1}{4\pi} \sum_{n=0}^{\infty} \int \frac{1}{\mathscr{r}} \frac{(\hat{\boldsymbol{r}} \cdot \boldsymbol{r}')^n}{n! c^n} \frac{\partial^n S}{\partial t^n}\bigg|_{t=t_0} d^3\boldsymbol{r}' \tag{41}$$

where we are dropping errors of relative order $1/r^2$ in the expansion of $r - \mathscr{r}$. Since we are already making errors of that order, there is no sense in keeping around the whole denominator $\frac{1}{\mathscr{r}}$, and we might as well approximate it by $\frac{1}{r}$, making the same order error. This can come outside of the integral, since it does not depend on $\boldsymbol{r}'$. Similarly, we can pull out the partial time derivatives. After the dust settles we have the promised Eq. (40).