UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2019

Problem Set 7 — SOLUTIONS

Due: Thursday, Apr. 11, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. **Resonant cavity modes**. Take what you've learned in the analysis of waveguides and apply it to a resonant cavity. Suppose we have a hollow rectangular box of side lengths $a \geq b \geq d$, and this box is made out of an excellent conductor. Solve Maxwell's equations subject to the appropriate boundary conditions at all the surfaces. Find the modes that are possible, which should now be labeled by three integers (l, m, n), and find the associated frequency ω_{lmn}. What is the general solution for $\boldsymbol{E}$ and $\boldsymbol{B}$ in one of these modes?

   Solution: We have to generalize the ansatz since we don't know the z dependence a priori. We can try $\boldsymbol{E} = \boldsymbol{E}_0(x, y, z)e^{-i\omega t}$ and $\boldsymbol{B} = \boldsymbol{B}_0(x, y, z)e^{-i\omega t}$. With this ansatz, the time dependence is gone from Maxwell's Eqs., becoming

$$\nabla \cdot \boldsymbol{E}_0 = 0 \,, \qquad \nabla \times \boldsymbol{E}_0 = i\omega \boldsymbol{B}_0 \,, \tag{1}$$

$$\nabla \cdot \boldsymbol{B}_0 = 0 \,, \qquad \nabla \times \boldsymbol{B}_0 = -\frac{i\omega}{c^2}\boldsymbol{E}_0 \,. \tag{2}$$

   We have to solve these equations subject to the boundary conditions $\boldsymbol{E}^{\parallel} = 0$, $B^{\perp} = 0$ on all surfaces. Once we have a solution for either $\boldsymbol{E}$ or $\boldsymbol{B}$, we can immediately find the other by taking a curl. Now take a curl of one of the curl equations and plug in the other Maxwell Eqs. to find

$$\nabla^2 \boldsymbol{E} = -\frac{\omega^2}{c^2}\boldsymbol{E} \,. \tag{3}$$

   This is just three separate, uncoupled copies of the Laplace equation. However the fields are coupled through the original Maxwell Eqs. Let's use separation of variables for $E_x(x, y, z) = X(x)Y(y)Z(z)$. Then we will find

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = -\frac{\omega^2}{c^2} \,. \tag{4}$$

   Therefore each term must be a constant, i.e. $d^2X/dx^2 = -k_x^2 X$ and similarly for Y and Z with k_y and k_z, satisfying $k_x^2 + k_y^2 + k_z^2 = -\omega^2/c^2$. Now we can use the boundary conditions to fix the constants in

$$E_x(x, y, z) = [\alpha_x \sin(k_x x) + \beta_x \cos(k_x x)][\alpha_y \sin(k_y y) + \beta_y \cos(k_y y)][\alpha_z \sin(k_z z) + \beta_z \cos(k_z z)] \,. \tag{5}$$

   Using $\boldsymbol{E}^{\parallel} = 0$ at $y = 0$ and $z = 0$ tells us that $\beta_y = \beta_z = 0$. Using this same B.C. at $y = b$ and $z = d$ tells us that $k_y = n\pi/b$ and $k_z = l\pi/d$ with integers n, l. If we apply the same approach to E_y and E_z, we will get (changing the names of constants!)

$$E_x(x, y, z) = [\alpha_x \sin(k_x x) + \beta_x \cos(k_x x)] \sin(k_y y) \sin(k_z z) \tag{6}$$

$$E_y(x, y, z) = \sin(k_x x)[\alpha_y \sin(k_y y) + \beta_y \cos(k_y y)] \sin(k_z z) \tag{7}$$

$$E_z(x, y, z) = \sin(k_x x) \sin(k_y y)[\alpha_z \sin(k_z z) + \beta_z \cos(k_z z)] \,, \tag{8}$$

   where $k_x = m\pi/a$ with m another integer.

We are still not done fixing the coefficients. Use $\nabla \cdot \boldsymbol{E} = 0$ to find that

$$\begin{aligned} 0 = k_x[\alpha_x \cos(k_x x) - \beta_x \sin(k_x x)] \sin(k_y y) \sin(k_z z) \\ + k_y \sin(k_x x)[\alpha_y \cos(k_y y) - \beta_y \sin(k_y y)] \sin(k_z z) \\ + k_z \sin(k_x x) \sin(k_y y)[\alpha_z \cos(k_z z) - \beta_z \sin(k_z z)] . \end{aligned} \tag{9}$$

Evaluating this at $x = 0$ we find that $k_x \alpha_x \sin(k_y y) \sin(k_z z) = 0$ for all y, z, so $\alpha_x = 0$. We similarly find $\alpha_y = \alpha_z = 0$. What remains is the condition

$$0 = (-\beta_x k_x - \beta_y k_y - \beta_z k_z) \sin(k_x x) \sin(k_y y) \sin(k_z z) , \tag{10}$$

so the amplitudes have to satisfy $\beta_x k_x + \beta_y k_y + \beta_z k_z = 0$.

To complete the solution, plug back in to $\nabla \times \boldsymbol{E} = i\omega \boldsymbol{B}$ and solve for $\boldsymbol{B}$ to find

$$B_x = \frac{-i}{\omega}(\beta_z k_y - \beta_y k_z)(\sin(k_x x) \cos(k_y y) \cos(k_z z)) \tag{11}$$

$$B_y = \frac{-i}{\omega}(\beta_x k_z - \beta_z k_x)(\cos(k_x x) \sin(k_y y) \cos(k_z z)) \tag{12}$$

$$B_z = \frac{-i}{\omega}(\beta_y k_x - \beta_x k_y)(\cos(k_x x) \cos(k_y y) \sin(k_z z)) , \tag{13}$$

where as before $k_x = m\pi/a, k_y = n\pi/b, k_z = l\pi/d$ for integers l, m, n, and with $\beta_x k_x + \beta_y k_y + \beta_z k_z = 0$. The modes have frequencies

$$\omega^2 = \omega_{lmn}^2 \equiv c^2(k_x^2 + k_y^2 + k_z^2) = c^2 \pi^2 \left[(m/a)^2 + (n/b)^2 + (l/d)^2\right] . \tag{14}$$

2. **Uniqueness of Lorenz gauge**. Suppose somebody hands you a $V, \boldsymbol{A}$ that solve Maxwell's equations in the potential formulation, and you check that they satisfy the Lorenz gauge condition $\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$. Can you perform a gauge transformation *generated* by a scalar function $\lambda(t, \boldsymbol{r})$ to a different gauge, but still satisfying the Lorenz gauge condition – what differential equation needs to be solved? How much freedom is there for λ that will take you between two different Lorenz gauges?

   Solution: A gauge transformation replaces the potentials $V, \boldsymbol{A}$ with new ones $V', \boldsymbol{A}'$ which are related by

$$V' = V - \frac{\partial \lambda}{\partial t} \tag{15}$$

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla \lambda . \tag{16}$$

   To find what gauge transformations enforce the Lorenz gauge condition, we need them to satisfy

$$0 = \nabla \cdot \boldsymbol{A}' + \frac{1}{c^2}\frac{\partial V'}{\partial t} \tag{17}$$

$$0 = \left(\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) + \left(\nabla^2 \lambda - \frac{1}{c^2}\frac{\partial^2 \lambda}{\partial t^2}\right) . \tag{18}$$

   The first term vanishes be cause we started in the Lorenz gauge. The second term says that any λ that satisfies the wave equation will preserve the Lorenz gauge condition.

3. What are the electric and magnetic fields that correspond to

$$V = 0 , \quad \boldsymbol{A} = \frac{-1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{\boldsymbol{r}} \, ? \tag{19}$$

   Find $V', \boldsymbol{A}'$ in another gauge via the gauge transformation function $\lambda = -(1/4\pi\epsilon_0)(qt/r)$. What is this new gauge?

Solution:

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}} \tag{20}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = 0\,. \tag{21}$$

Evidently this is the field of a stationary point charge at the origin. Now applying the gauge transformation,

$$V' = V - \frac{\partial \lambda}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{22}$$

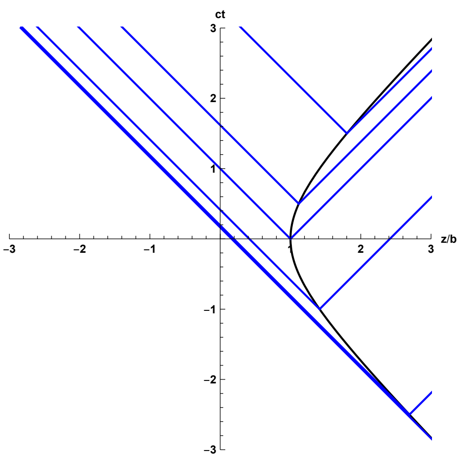
$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\lambda = 0\,. \tag{23}$$

These were our earlier starting potentials in electrostatics. This is now the Coulomb gauge. It also happens to be in Lorenz gauge.

4. **Practice with finding retarded time**. Suppose a particle follows the hyperbolic trajectory $\boldsymbol{\xi}(t) = \hat{\boldsymbol{z}}\sqrt{b^2 + c^2t^2}$.

   (a) Draw the space-time diagram with $z - t$ axes to show this particle's motion. Draw the light signals that would emanate from the particle.

   Solution: The particle follows the black trajectory, and the blue lines denote light rays.



   (b) Notice that there are some regions in spacetime that do not know about the existence of the particle! What points in the (t, z) plane (setting $x = y = 0$) haven't yet received a signal from the particle?

   Solution: Points where $z < -ct$ have not yet seen the particle.

   (c) From the implicit equation for the definition of retarded time, show that $t_r(t, \boldsymbol{r})$ can be found by solving a quadratic equation.

   Solution: We want to solve for t_r (which is really the function $t_r(t, \boldsymbol{r})$) that satisfies the implicit equation

$$c^2(t - t_r)^2 = x^2 + y^2 + (z - \xi(t_r))^2 = x^2 + y^2 + z^2 + (b^2 + c^2t_r^2) - 2z\sqrt{b^2 + c^2t^2} \tag{24}$$

   This does not obviously have a solution by radicals, but there is a $+c^2t_r^2$ on both the left and right hand sides that cancel. After isolating the radical above and squaring,

$$\left[c^2t^2 - 2c^2tt_r - x^2 - y^2 - z^2 - b^2\right]^2 = 4z^2(b^2 + c^2t_r^2)\,. \tag{25}$$

   This is quadratic in t_r on both sides so we can find a solution by radicals.

(d) Actually solve it in the simpler case where your position is along the z axis, $\hat{\boldsymbol{r}} = z\hat{\boldsymbol{z}}$, and in the appropriate region so that you've received a signal from the particle. You can be either above or below the particle; find the solution in both cases.

Solution: When $x = y = 0$, the solutions are

$$t_r = \frac{-b^2 + (ct \pm z)^2}{2c(ct \pm z)} \,. \tag{26}$$

When your z coordinate is greater than the particle, your position is "inside" the hyperbola in the above figure. There the function t_r should be constant on "right"-going light rays, i.e. those with positive slope in the figure; t_r can remain constant when both z and t increase. Therefore to the right is the negative root. Meanwhile if your z coordinate is less than the particle's, your position is "outside" the hyperbola in the figure. Then t_r should be constant on "left"-going light rays, i.e. those with negative slope; thus we want the positive root.

5. Start from the Liénard-Wiechart scalar potential for a charge in uniform linear motion,

$$V = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(ct - \boldsymbol{r}\cdot\boldsymbol{v}/c)^2 + (1 - v^2/c^2)(r^2 - c^2t^2)}} \,. \tag{27}$$

Show that this can be rewritten in terms of $\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{v}t$, the "instantaneous" separation, as

$$V = \frac{q}{4\pi\epsilon_0} \frac{1}{R\sqrt{1 - (\sin^2\theta)v^2/c^2}} \,, \tag{28}$$

where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{R}$.

Solution: Call the argument under the square root in Eq. (27)

$$I \equiv (ct - \boldsymbol{r}\cdot\boldsymbol{v}/c)^2 + (1 - v^2/c^2)(r^2 - c^2t^2) \tag{29}$$
$$I = c^2t^2 + (\boldsymbol{r}\cdot\boldsymbol{v})^2/c^2 - 2t(\boldsymbol{r}\cdot\boldsymbol{v}) + r^2 - c^2t^2 - r^2v^2/c^2 + t^2v^2 \tag{30}$$
$$I = (\boldsymbol{r}\cdot\boldsymbol{v})^2/c^2 - 2(\boldsymbol{r}\cdot\boldsymbol{v}t) + r^2(1 - v^2/c^2) + (vt)^2 \tag{31}$$

Everywhere you see $\boldsymbol{v}t$, replace it with $\boldsymbol{v}t = \boldsymbol{r} - \boldsymbol{R}$. This gives

$$I = (\boldsymbol{r}\cdot\boldsymbol{v})^2/c^2 - 2(\boldsymbol{r}\cdot(\boldsymbol{r} - \boldsymbol{R})) + r^2(1 - v^2/c^2) + (\boldsymbol{r} - \boldsymbol{R})^2 \tag{32}$$
$$I = (\boldsymbol{r}\cdot\boldsymbol{v})^2/c^2 - 2r^2 + 2\boldsymbol{r}\cdot\boldsymbol{R} + r^2(1 - v^2/c^2) + r^2 + R^2 - 2\boldsymbol{r}\cdot\boldsymbol{R} \tag{33}$$
$$I = (\boldsymbol{r}\cdot\boldsymbol{v})^2/c^2 - r^2v^2/c^2 + R^2 \tag{34}$$

Now notice that

$$(\boldsymbol{r}\cdot\boldsymbol{v})^2 - r^2v^2 = ((\boldsymbol{R} + \boldsymbol{v}t)\cdot\boldsymbol{v})^2 - (\boldsymbol{R} + \boldsymbol{v}t)^2v^2 \tag{35}$$
$$= (\boldsymbol{R}\cdot\boldsymbol{v})^2 + v^4t^2 + 2(\boldsymbol{R}\cdot\boldsymbol{v})v^2t - R^2v^2 - 2(\boldsymbol{R}\cdot\boldsymbol{v})tv^2 - v^4t^2 \tag{36}$$
$$= (\boldsymbol{R}\cdot\boldsymbol{v})^2 - R^2v^2 = R^2v^2(\cos^2\theta - 1) = -(\sin^2\theta)R^2v^2 \tag{37}$$

Therefore

$$I = R^2(1 - (\sin^2\theta)v^2/c^2) \tag{38}$$

and so we have the desired result,

$$V = \frac{q}{4\pi\epsilon_0} \frac{1}{R\sqrt{1 - (\sin^2\theta)v^2/c^2}} \,. \tag{39}$$

6. **Energy flux in the field of a uniformly moving charge**.

(a) For a charge in uniform linear motion, what is the Poynting vector? (I think in lecture I erroneously claimed it vanished!).

Solution: In uniform linear motion, we previously found that

$$\boldsymbol{B} = \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{E}\,. \tag{40}$$

When we plug this into the Poynting vector,

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B} = \frac{1}{\mu_0 c^2}\left[\boldsymbol{E} \times (\boldsymbol{v} \times \boldsymbol{E})\right] \tag{41}$$

$$\boldsymbol{S} = \epsilon_0\left[E^2\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{E})\boldsymbol{E}\right] \tag{42}$$

which does not vanish.

(b) Suppose the charge's velocity is purely in the $\hat{\boldsymbol{z}}$ direction, $\boldsymbol{v} = v_z\hat{\boldsymbol{z}}$. Integrate the energy flux through the entire $z = 0$ plane. What is the resulting dE/dt, as a function of the particle's position d along the $\boldsymbol{z}$ axis?

Solution: We want the integral

$$\frac{dE}{dt} = \int \boldsymbol{S} \cdot \mathrm{da} = \int_0^{2\pi}\int_0^{\infty} \mathrm{S_z}\rho\mathrm{d}\rho\mathrm{d}\phi \tag{43}$$

where $\rho = \sqrt{x^2 + y^2}$ is the distance along the plane, and ϕ is the angle in the plane. Nothing depends on ϕ so we just pick up a factor of 2π. Now we have to compute S_z from $\boldsymbol{v} = v_z\hat{z}$ and $\boldsymbol{E}$,

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{\gamma^2}\frac{\hat{\boldsymbol{R}}}{R^2(1 - (\sin^2\theta)v^2/c^2)^{3/2}} \tag{44}$$

where $\gamma^2 = 1/(1 - v^2/c^2)$. The norm of this is easy because $|\hat{R}| = 1$, and then we also need the component E_z for both $\boldsymbol{v} \cdot \boldsymbol{E}$ and E_z in S_z. Suppose we are at position ρ when the particle is at distance d along the axis. Then there is a right triangle with sides (d, ρ, R) where $R^2 = \rho^2 + d^2$ and angle θ where $\tan\theta = \rho/d$. At this point ρ, the value of S_z is

$$S_z = \epsilon_0[E^2 v - E^2 v\cos^2\theta] = \epsilon_0 E^2 v\sin^2\theta\,. \tag{45}$$

One factor of $\cos\theta$ came from $\boldsymbol{E} \cdot \boldsymbol{v}$ and the other from the z component of E.

Now our integral is

$$\frac{dE}{dt} = 2\pi\epsilon_0 v\int_0^{\infty} E^2\cos^2\theta\rho\mathrm{d}\rho = 2\pi\epsilon_0\mathrm{v}\left(\frac{\mathrm{q}}{4\pi\epsilon_0}\frac{1}{\gamma^2}\right)^2\int_0^{\infty}\frac{\sin^2\theta}{\mathrm{R}^4(1 - (\sin^2\theta)\mathrm{v}^2/\mathrm{c}^2)^3}\rho\mathrm{d}\rho\,. \tag{46}$$

Here R and θ all depend on ρ. Let's use θ instead as an integration variable, $\rho = d\tan\theta$ so $\mathrm{d}\rho = \mathrm{d}/\cos^2\theta\mathrm{d}\theta$ and $1/R = \cos\theta/d$. Now we have

$$\frac{dE}{dt} = 2\pi\epsilon_0 v\left(\frac{q}{4\pi\epsilon_0}\frac{1}{\gamma^2}\right)^2\frac{1}{d^2}\int_0^{\pi/2}\frac{\sin^3\cos\theta\mathrm{d}\theta}{(1 - (\sin^2\theta)v^2/c^2)^3}\,. \tag{47}$$

One final change of variables: let $u = \sin^2\theta$ and note that we already have $\mathrm{du} = 2\sin\theta\cos\theta\mathrm{d}\theta$ in the integrand. This gives

$$\frac{dE}{dt} = \frac{vq^2}{16\pi\epsilon_0 d^2\gamma^4}\int_0^1\frac{u\mathrm{du}}{(1 - uv^2c^2)^3} = \frac{vq^2}{32\pi\epsilon_0 d^2}\,. \tag{48}$$