UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2019

Problem Set 6 — SOLUTIONS

Due: Wednesday, Apr. 3, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. **Reflection with horizontal polarization** (long). In lecture we went through the derivation of reflection where the electric field is "vertically" polarized, i.e. with the $\boldsymbol{E}$ field lying in the $x-z$ plane of incidence. Redo the calculation but with the horizontal polarization, i.e. with $\boldsymbol{E} \propto \hat{y}$.

   (a) Write down the four boundary conditions, evaluated with the appropriate parallel/perpendicular electric and magnetic fields, in terms of the angles θ_I, θ_T etc.
   Solution: We will have the fields

$$\tilde{\boldsymbol{E}}_I = \tilde{E}_{0I} e^{i(\boldsymbol{k}_I \cdot \boldsymbol{r} - \omega t)} \hat{\boldsymbol{y}} \tag{1}$$

$$\tilde{\boldsymbol{B}}_I = \frac{1}{v_1} \tilde{E}_{0I} e^{i(\boldsymbol{k}_I \cdot \boldsymbol{r} - \omega t)} (-\cos\theta_I \hat{\boldsymbol{x}} + \sin\theta_I \hat{\boldsymbol{z}}) \tag{2}$$

$$\tilde{\boldsymbol{E}}_R = \tilde{E}_{0R} e^{i(\boldsymbol{k}_R \cdot \boldsymbol{r} - \omega t)} \hat{\boldsymbol{y}} \tag{3}$$

$$\tilde{\boldsymbol{B}}_R = \frac{1}{v_1} \tilde{E}_{0R} e^{i(\boldsymbol{k}_R \cdot \boldsymbol{r} - \omega t)} (\cos\theta_I \hat{\boldsymbol{x}} + \sin\theta_I \hat{\boldsymbol{z}}) \tag{4}$$

$$\tilde{\boldsymbol{E}}_T = \tilde{E}_{0T} e^{i(\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega t)} \hat{\boldsymbol{y}} \tag{5}$$

$$\tilde{\boldsymbol{B}}_T = \frac{1}{v_2} \tilde{E}_{0T} e^{i(\boldsymbol{k}_T \cdot \boldsymbol{r} - \omega t)} (-\cos\theta_T \hat{\boldsymbol{x}} + \sin\theta_T \hat{\boldsymbol{z}}) \tag{6}$$

   where we already know that $\theta_I = \theta_R$ and the law of refraction ($\sin\theta_T / \sin\theta_I = v_2/v_1$). We can impose the four boundary conditions

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \tag{7a}$$

$$\boldsymbol{E}_1^{\parallel} = \boldsymbol{E}_2^{\parallel} \tag{7b}$$

$$B_1^{\perp} = B_2^{\perp} \tag{7c}$$

$$\frac{1}{\mu_1} \boldsymbol{B}_1^{\parallel} = \frac{1}{\mu_2} \boldsymbol{B}_2^{\parallel} \tag{7d}$$

   The first is automatically satisfied. The second tells us $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$. The third tells us the same thing, after using the law of refraction (or alternatively, this is where the law comes from). The fourth tells us

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \left(\frac{\mu_1 v_1 \cos\theta_T}{\mu_2 v_2 \cos\theta_I} \right) \tilde{E}_{0T} \,. \tag{8}$$

   (b) Solve the resulting linear system for the ratios $\tilde{\boldsymbol{E}}_{0R}/\tilde{\boldsymbol{E}}_{0I}$ and $\tilde{\boldsymbol{E}}_{0T}/\tilde{\boldsymbol{E}}_{0I}$, in terms of the earlier variables $\alpha \equiv \cos\theta_T / \cos\theta_I$ and $\beta \equiv \mu_1 v_1 / \mu_2 v_2$.
   Solution: The solution to the system is

$$\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{1 - \alpha\beta}{1 + \alpha\beta} \tag{9}$$

$$\frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = \frac{2}{1 + \alpha\beta} \,. \tag{10}$$

(c) Give an equation for Brewster's angle θ_B where there is no (horizontal) reflection. Assuming that $\mu_1 \approx \mu_2$ to simplify, what do you find for the no-reflection condition?

Solution: No reflection occurs when $\alpha\beta = 1$. In terms of angles, this happens when

$$\alpha = \frac{\sqrt{1 - \sin^2\theta v_2^2/v_1^2}}{\cos\theta} = \frac{1}{\beta} = \frac{\mu_2 v_2}{\mu_1 v_1} \tag{11}$$

or

$$1 = \left(\frac{v_2}{v_1}\right)^2 [\sin^2\theta + \frac{\mu_2^1}{\mu_1^2}\cos^2\theta]\,. \tag{12}$$

If $\mu_1 \approx \mu_2$, this only happens when $v_1 \approx v_2$, but that just means the two media are optically identical, so there wouldn't be any reflection.

(d) Check that the reflection and transmission coefficients add up to 1 (recall that the transmission coefficient is the ratio of intensities, rather than the square of the ratio of electric fields).

Solution: The reflection and transmission coefficients are

$$R = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2, \qquad T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\alpha\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2. \tag{13}$$

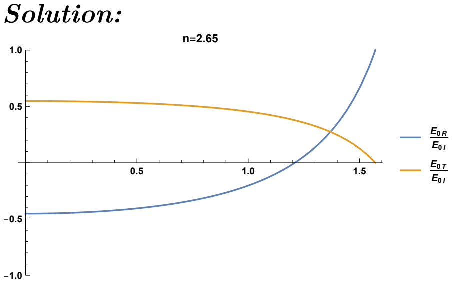
The sum is indeed 1,

$$R + T = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 + \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2 = \frac{1 - 2\alpha\beta + \alpha^2\beta^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = 1\,. \tag{14}$$

2. Silicon carbide has an index of refraction of $n = 2.65$.

(a) Plot the ratios E_{0R}/E_{0I} and E_{0T}/E_{0I} as a function of θ_I, for the interface between SiC and air (assuming $\mu_1 = \mu_2 = \mu_0$).

Solution:



(b) What are the values for the two amplitude ratios at normal incidence?

Solution: At $\theta = 0$, $E_{0R}/E_{0I} \approx -0.45$ and $E_{0T}/E_{0I} \approx +0.55$.

(c) What is Brewster's angle?

Solution: $\theta_B \approx 69.3^\circ \approx 1.21$ rad.

(d) What is the "crossover" angle, where the reflection and transmission amplitudes are equal?

Solution: $\theta_x \approx 78.5^\circ \approx 1.37$ rad.

3. (a) Suppose you embedded some free charge in a piece of glass. About how long would it take for the charge to flow to the surface?

Solution: Using $\tau = \epsilon/\sigma$, with $\epsilon \approx \epsilon_0 n^2$. For glass, $n \approx 1.5$. Meanwhile the conductivity for glass is approximately $\sigma = 1/\rho \approx 10^{-12}\Omega$ m. Plugging in the numbers we get $\tau \approx 20$ s.

(b) Silver is an excellent conductor, but it's expensive. Suppose you were designing a microwave experiment to operate at a frequency of 10^{10}Hz. How thick would you make the silver coatings?

Solution: For silver, $\rho \approx 1.59 \times 10^{-8}$ and $\epsilon \approx \epsilon_0$. Compare $\omega\epsilon \approx 0.56$ with $\sigma = 1/\rho = 6.25 \times 10^7$ which is much larger than $\omega\epsilon$. So we can approximate

$$d = \frac{1}{\kappa} \approx \sqrt{\frac{2}{\omega\sigma\mu}} = 6.4 \times 10^{-4} \text{ mm}. \tag{15}$$

You only need a few times this skin depth.

(c) Find the wavelength and propagation speed in copper for radio waves at 1 MHz. Compare the corresponding values in air (or vacuum).

Solution: Once again let's compare $\sigma = 1/(1.68 \times 10^{-8}) = 6 \times 10^7$ against $\omega\epsilon_0 = 6 \times 10^{-5}$. In this case since $\sigma \gg \omega\epsilon$, we can approximate $k \approx \sqrt{\omega\sigma\mu/2}$ and thus

$$\lambda = 2\pi\sqrt{\frac{2}{\omega\sigma\mu_0}} = 0.4 \text{ mm}. \tag{16}$$

The propagation speed is $v = \omega/k = \omega\lambda/2\pi = 400$ m/s. This is extremely small compared to $c = 3 \times 10^8$ m/s. The wavelength in vacuum would be $\lambda = c2\pi/\omega = 300$ m.

4. Griffiths 9.19a-c (asymptotic limits of skin depth for very good/bad conductors; and magnetic field lag and strength in a good conductor). For part c, in SI units, the ratio of B/E is not dimensionless, so it's silly to look at the numerical value by itself. Rather, find the ratio of B/E in a good conductor, and compare it to the ratio in vacuum.

Solution:

(a) We use the expansion $\sqrt{1+x} \approx 1 + \frac{1}{2}x + \mathcal{O}(x^2)$ when $x \ll 1$. Then it is some simple algebra to find

$$\kappa \approx \omega\sqrt{\frac{\epsilon\mu}{2}}\left[1 + \frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2 - 1\right]^{1/2} \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}. \tag{17}$$

Now the skin depth comes from κ via $d = 1/\kappa \approx \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$.

For water, $\sigma \approx 1/(2.5 \times 10^5)$, $\mu \approx \mu_0$, and $\epsilon \approx 80\epsilon_0$. Plugging these in we find $d \approx 10^4$ m.

(b) In a very good conductor, we can drop the 1 inside the innermost square root of both k and κ, because $(\sigma\epsilon/\omega)^2 \gg 1$. After taking the square root we can drop the other 1 for the same reason. Therefore we found that $k \approx \kappa$, so the skin depth satisfied $d = 1/\kappa = 1/k = \lambda/2\pi$. For a metal with the given values, we have $d \approx 13$ nm. The opacity comes from the fact that we almost always encounter metals that are thicker than this skin depth, so they are not transmitting any light.

(c) The phase lag satisfies $\tan\phi = \kappa/k \approx 1$ for a very good conductor, so $\phi \approx 45°$. Inside of this typical metal, we would find $B_0/E_0 \approx \sqrt{\sigma\mu/\omega} \approx 10^{-7}$ s/m. This is to be compared with the same ratio in vacuum, which is one over the speed of light, 3×10^{-9} s/m. The ratio of $(B_0/E_0)_{metal}/(B_0/E_0)_{vac} \approx 30$.

5. Griffiths 9.22a-b (phase and group velocities in deep water waves, and quantum mechanics). Note that Griffiths writes "wave velocity" for what everyone calls the *phase* velocity.

Solution:

(a) Deep water waves satisfy

$$v_{ph} = \frac{\omega}{k} \propto \sqrt{\lambda}. \tag{18}$$

Therefore we have $\omega = Ck^{1/2}$ for some constant C. Then we can find the group velocity as

$$v_g = \frac{d\omega}{dk} = \frac{C}{2\sqrt{k}} = \frac{1}{2}\frac{\omega}{k} = \frac{1}{2}v_{ph}. \tag{19}$$

(b) In the phase factor, we identify $kx - \omega t = px - Et$. So we find $k = p$ and $\omega = E = p^2/2m = k^2/2m$. From this dispersion relationship we find the phase and group velocities,

$$v_{ph} = \frac{\omega}{k} = \frac{k}{2\pi} \tag{20}$$

$$v_g = \frac{d\omega}{dk} = \frac{k}{\pi} = 2v_{ph}. \tag{21}$$

The classical particle speed is $v = p/m = k/m = v_g$.

6. Consider a rectangular waveguide with sides 3.42 cm $\times$ 1.515 cm. What TE modes will propagate in this waveguide, if the driving frequency is 1.13×10^{10}Hz? Suppose you want to excite only one TE mode. What range of frequencies could you use? What are the corresponding wavelengths of those frequencies when they are in free space?

Solution: There are four valid modes: $(0,1), (1,0), (1,1), (2,0)$. Their corresponding frequencies are 9.9×10^9 Hz, 4.4×10^9 Hz, 1.1×10^{10} Hz, and 8.8×10^9 Hz. To get just one mode (the $(1,0)$ mode), use any frequency in the range 4.4×10^9 Hz $< \nu < 8.8 \times 10^9$ Hz. The range of wavelengths that this corresponds to is $\lambda = c/\nu$, 3.4 cm $< \lambda <$ 6.8 cm.