UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2019

Problem Set 4 — SOLUTIONS

Due: Friday, Mar. 8, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. Griffiths problem 9.9a-b (The real $\boldsymbol{E}$ and $\boldsymbol{B}$ fields due for two example monochromatic plane waves).

   Solution:

   (a) $\boldsymbol{k} = (-\omega/c)\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$. From these we can compute $\boldsymbol{k}\cdot\boldsymbol{r} = (-\omega/c)x$ (recall that $\boldsymbol{r} = x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+z\hat{\boldsymbol{z}}$). We can also compute $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}} = \hat{\boldsymbol{y}}$. Now we can write down the real fields,

   $$\boldsymbol{E} = E_0 \cos(\frac{\omega}{c}x + \omega t)\hat{\boldsymbol{z}}\,, \qquad \boldsymbol{B} = \frac{E_0}{c}\cos(\frac{\omega}{c}x + \omega t)\hat{\boldsymbol{y}}\,. \tag{1}$$

   (b) $\boldsymbol{k}$ should have direction $(1, 1, 1)$ but magnitude ω/c. The normalization constant is found very easily, and $\boldsymbol{k} = (\omega/c)(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}})/\sqrt{3}$. To find $\hat{\boldsymbol{n}}$, write it as $\hat{\boldsymbol{n}} = a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{z}}$, where $a^2 + b^2 = 1$. Now require that $\hat{\boldsymbol{n}} \cdot \boldsymbol{k} = 0$ and find $\hat{\boldsymbol{n}} = (\hat{x} - \hat{z})/\sqrt{2}$. Now we can compute $\boldsymbol{k} \cdot \boldsymbol{r} = (\omega/\sqrt{3}c)(x + y + z)$ and $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}} = (-\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}})/\sqrt{2}$. Combining we get

   $$\boldsymbol{E} = E_0 \cos\left[\frac{\omega}{\sqrt{3}c}(x + y + z) - \omega t\right](\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}})/\sqrt{2}\,, \tag{2}$$

   $$\boldsymbol{B} = \frac{E_0}{c}\cos\left[\frac{\omega}{\sqrt{3}c}(x + y + z) - \omega t\right](-\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}})/\sqrt{6}\,. \tag{3}$$

2. Griffiths problem 9.12 (The Maxwell stress tensor due to a monochromatic plane wave traveling in the z direction).

   Solution: Starting from

   $$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)\,, \tag{4}$$

   we will plug in a plane wave traveling in the z direction, linearly polarized in the x direction, namely

   $$\boldsymbol{E} = E_0\cos(kz - \omega t)\hat{\boldsymbol{x}}\,, \qquad \boldsymbol{B} = \frac{E_0}{c}\cos(kz - \omega t)\hat{\boldsymbol{y}}\,. \tag{5}$$

   The only non-vanishing components in this case are the diagonal ones, which take on values

   $$T_{xx} = \epsilon_0\left(E_xE_x - \frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B^2\right) = \frac{1}{2}\left(\epsilon_0E^2 - \frac{1}{\mu_0}B^2\right) = 0\,, \tag{6}$$

   $$T_{yy} = \epsilon_0\left(-\frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(B_yB_y - \frac{1}{2}B^2\right) = \frac{1}{2}\left(-\epsilon_0E^2 + \frac{1}{\mu_0}B^2\right) = 0\,, \tag{7}$$

   $$T_{zz} = \epsilon_0\left(-\frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(-\frac{1}{2}B^2\right) = -\epsilon_0E_0^2\cos^2(kz - \omega t) = -u\,. \tag{8}$$

   This is consistent with the fact that T_{ij} quantifies the amount of $i-$momentum being transported in the j direction. In this case we have momentum flux density = energy density.

3. **Stress in index notation**. Suppose you have a monochromatic plane wave where the direction and k-number are given by the vector $\boldsymbol{k}$, or k_i in index notation; and this wave is linearly polarized with unit polarization vector $\hat{\boldsymbol{e}}$, or $\hat{e}_i$ in index notation, where $\boldsymbol{k} \cdot \hat{\boldsymbol{e}} = 0 = k_i \hat{e}_i$. Find the Maxwell stress tensor T_{ij} in index notation, in terms of the above quantities.

Solution: Translating the linearly-polarized plane wave solution into index notation, we have

$$E_i = \hat{e}_i E_0 \cos(k_j x^j - \omega t)\,, \tag{9}$$

$$B_i = \epsilon_{ijk} \hat{k}^j \hat{e}^k \frac{E_0}{c} \cos(k_j x^j - \omega t)\,. \tag{10}$$

(We must be careful since k is being used as both an index and the name of a vector). Now we need to insert this into

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)\,, \tag{11}$$

and make use of the identity for the product of two epsilon tensors, which can be compactly written as

$$\epsilon^{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta^i_l & \delta^i_m & \delta^i_n \\ \delta^j_l & \delta^j_m & \delta^j_n \\ \delta^k_l & \delta^k_m & \delta^k_n \end{vmatrix} = \delta^i_l \left(\delta^j_m \delta^k_n - \delta^j_n \delta^k_m \right) - \delta^i_m \left(\delta^j_l \delta^k_n - \delta^j_n \delta^k_l \right) + \delta^i_n \left(\delta^j_l \delta^k_m - \delta^j_m \delta^k_l \right). \tag{12}$$

First, let's compute both E^2 and B^2. The first is very simple,

$$E^2 = E_i E^i = E_0^2 \hat{e}_i \hat{e}^i \cos^2 S = E_0^2 \cos^2 S\,, \tag{13}$$

where $S \equiv k_j x^j - \omega t$ and we have used the fact that $\hat{e}_i \hat{e}^i = \hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{e}} = 1$. Now for the more complicated B^2, where we have to make use of Eq. (12),

$$B^2 = B_i B^i = \frac{E_0^2}{c^2} \cos^2 S\, \epsilon_{ijk} \epsilon^i{}_{lm} \hat{k}^j \hat{e}^k \hat{k}^l \hat{e}^m \tag{14}$$

$$B^2 = \frac{E_0^2}{c^2} \cos^2 S \left((\hat{k}_j \hat{k}^j)(\hat{e}_k \hat{e}^k) - (\hat{k}_j \hat{e}^j)(\hat{k}_k \hat{e}^k) \right) \tag{15}$$

$$B^2 = \frac{E_0^2}{c^2} \cos^2 S \tag{16}$$

where have also made use of $\hat{e}_i \hat{e}^i = \hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{e}} = 1$ and $\hat{e}_i \hat{k}^i = \hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{k}} = 0$.

Finally we can assemble the Maxwell stress tensor,

$$T_{ij} = \epsilon_0 E_0^2 \cos^2 S \left[\left(\hat{e}_i \hat{e}_j - \frac{1}{2} \delta_{ij} \right) + \left(\epsilon_{ikl} \epsilon_{jmn} \hat{k}^k \hat{e}^l \hat{k}^m \hat{e}^n - \frac{1}{2} \delta_{ij} \right) \right]. \tag{17}$$

Now we need to expand out the somewhat complicated expression $\epsilon_{ikl}\epsilon_{jmn} \hat{k}^k \hat{e}^l \hat{k}^m \hat{e}^n$. After using Eq. (12), contracting the δ tensors where possible, and using $\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{e}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1$ and $\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{k}} = 0$, we arrive at

$$\epsilon_{ikl}\epsilon_{jmn} \hat{k}^k \hat{e}^l \hat{k}^m \hat{e}^n = \delta_{ij} - \hat{e}_i \hat{e}_j - \hat{k}_i \hat{k}_j\,. \tag{18}$$

Plugging this back in we finally get

$$T_{ij} = \epsilon_0 E_0^2 \cos^2 S\, k_i k_j = -u\, k_i k_j\,. \tag{19}$$

4. **Standing electromagnetic wave**. Suppose we superimpose the following two complex, monochromatic waves, both with frequency ω: one traveling in the $\hat{z}$ direction, and the electric field is polarized in $\hat{x}$ with amplitude $\tilde{E}_0$; and the second is traveling in the $-\hat{z}$ direction, with electric field polarized in $-\hat{x}$ with the same amplitude $\tilde{E}_0$.

   (a) Write down the complex $\tilde{\boldsymbol{E}}, \tilde{\boldsymbol{B}}$ fields for each of the two waves separately.
   Solution: We will denote the left-moving and right-moving fields as $\tilde{\boldsymbol{E}}_L, \tilde{\boldsymbol{E}}_R$, etc.

$$\tilde{\boldsymbol{E}}_R = +\tilde{E}_0 \exp[i(kz - \omega t)]\hat{\boldsymbol{x}}\,, \tag{20}$$

$$\tilde{\boldsymbol{B}}_R = +\frac{1}{c}\tilde{E}_0 \exp[i(kz - \omega t)]\hat{\boldsymbol{y}}\,, \tag{21}$$

$$\tilde{\boldsymbol{E}}_L = -\tilde{E}_0 \exp[i(-kz - \omega t)]\hat{\boldsymbol{x}}\,, \tag{22}$$

$$\tilde{\boldsymbol{B}}_L = +\frac{1}{c}\tilde{E}_0 \exp[i(-kz - \omega t)]\hat{\boldsymbol{y}}\,. \tag{23}$$

   (b) Now superimpose them and find the *real* field that arises from the superposition.
   Solution: The physical fields are $\boldsymbol{E} = \mathrm{Re}[\tilde{\boldsymbol{E}}_L + \tilde{\boldsymbol{E}}_R]$ and $\boldsymbol{B} = \mathrm{Re}[\tilde{\boldsymbol{B}}_L + \tilde{\boldsymbol{B}}_R]$. This gives

$$\boldsymbol{E} = 2E_0 \sin(kz)\sin(\omega t)\hat{\boldsymbol{x}}\,, \tag{24}$$

$$\boldsymbol{B} = \frac{2}{c}E_0 \cos(kz)\cos(\omega t)\hat{\boldsymbol{y}}\,. \tag{25}$$

   (c) Find the part of the period-averaged energy density due just to the electric field and comment on its pattern in space.
   Solution: We are going to compute $\langle u_E\rangle = \langle \frac{1}{2}\epsilon_0 E^2\rangle$,

$$\langle \frac{1}{2}\epsilon_0 E^2\rangle = \frac{1}{T}\int_0^T \frac{1}{2}\epsilon_0(2E_0\sin(kz)\sin(\omega t))^2 dt \tag{26}$$

$$\langle \frac{1}{2}\epsilon_0 E^2\rangle = \epsilon_0 E_0^2 \sin^2(kz)\,, \tag{27}$$

   where the period is $T = 2\pi/\omega$. Notice that this is spatially modulated.

   (d) Now examine the total period-averaged energy density $\langle u\rangle$ (due to both the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields), and again comment on its pattern in space.
   Solution: Repeating for the magnetic field we find $\langle u_B\rangle = \epsilon_0 E_0^2\cos^2(kz)$, of the same amplitude and also spatially modulated, but with a different phase. Thus the sum is not spatially modulated: $\langle u\rangle = \epsilon_0 E_0^2$.

   (e) Finally, find the period-averaged Poynting vector, $\langle \boldsymbol{S}\rangle$.
   Solution: Computing the real Poynting vector we find

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} = \frac{4}{\mu_0 c}E_0^2\sin(kz)\sin(\omega t)\cos(kz)\cos(\omega t)\hat{\boldsymbol{z}}\,. \tag{28}$$

   Now when we period average we need the integral

$$\frac{1}{T}\int_0^T \sin(\omega t)\cos(\omega t)dt = 0\,. \tag{29}$$

   Therefore $\langle \boldsymbol{S}\rangle = 0$, and there is no energy flux.

5. **Getting no transmission at normal incidence**.

   (a) What is the factor β (appearing in reflection/transmission at normal incidence) in terms of just ϵ's and μ's of two media at an interface?
   Solution: The reflection and transmission amplitudes are all determined by the combination

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \sqrt{\frac{\mu_1\epsilon_2}{\mu_2\epsilon_1}}\,. \tag{30}$$

(b) In terms of ϵ's and μ's, how do we get no transmission from medium 1 into medium 2 at normal incidence? What property of a medium could cause this?

Solution: The transmission amplitude will vanish in the limit $\beta \to \infty$. This can be accomplished by taking $\mu_2 \to 0$. This happens in a superconductor.

(c) What does no transmission do to the reflected fields, $\tilde{\boldsymbol{E}}_R$ and $\tilde{\boldsymbol{B}}_R$?

Solution: The reflected fields will satisfy

$$\tilde{E}_{0R} = \frac{1-\beta}{1+\beta}\tilde{E}_{0I} \to -\tilde{E}_{0I} \tag{31}$$

$$\tilde{B}_{0R} \to -\tilde{B}_{0I}\,. \tag{32}$$

That is, they will have the same amplitude but opposite sign as the incident fields.

6. Griffiths problem 9.14 (Show that reflected and transmitted waves at normal incidence must have same polarization).

Solution: Let the incident wave be polarized along $\hat{\boldsymbol{x}}$, and the reflected and transmitted polarization vectors be

$$\hat{\boldsymbol{n}}_R = \cos\theta_R\hat{\boldsymbol{x}} + \sin\theta_R\hat{\boldsymbol{y}}\,, \tag{33}$$

$$\hat{\boldsymbol{n}}_T = \cos\theta_T\hat{\boldsymbol{x}} + \sin\theta_T\hat{\boldsymbol{y}}\,. \tag{34}$$

The magnetic fields will now be along $\tilde{\boldsymbol{B}}_R \propto -\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}_R$ and $\tilde{\boldsymbol{B}}_T \propto +\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}_T$. Now let us analyze the boundary conditions of equality of the electric and magnetic fields at the interface,

$$\tilde{E}_{0I}\hat{\boldsymbol{x}} + \tilde{E}_{0R}\hat{\boldsymbol{n}}_R = \tilde{E}_{0T}\hat{\boldsymbol{n}}_T\,, \tag{35}$$

$$\tilde{E}_{0I}\hat{\boldsymbol{x}} - \tilde{E}_{0R}(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}_R) = \beta\tilde{E}_{0T}(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}_T)\,. \tag{36}$$

The y component of Eq. (35) says

$$\tilde{E}_{0R}\sin\theta_R = \tilde{E}_{0T}\sin\theta_T\,, \tag{37}$$

and the x component of Eq. (36) says

$$\tilde{E}_{0R}\sin\theta_R = -\beta\tilde{E}_{0T}\sin\theta_T\,, \tag{38}$$

These equations can only be simultaneously solved when $\sin\theta_R = \sin\theta_T = 0$ or $\theta_R = \theta_T = 0$.