UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2019

Problem Set 3 — SOLUTIONS

Due: Thursday, Feb. 28, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. Griffiths problem 8.5a-d (Infinite parallel-plate capacitor's stress tensor, force per unit area, momentum flux, recoil per unit area).

   Solution:

   (a) There is no magnetic field, and the electric field is $E_z = -\sigma/\epsilon_0$ with other components vanishing. This gives the stress tensor

   $$\boldsymbol{T} = \frac{\sigma^2}{2\epsilon_0}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{1}$$

   (b) We are interested in the integrand of $\boldsymbol{F} = \oint \boldsymbol{T}\cdot d\boldsymbol{a}$, over a surface that encloses the top plate. Let the surface be a rectangle and infinitesimally thin. The only contribution to the integral is from the bottom face of this surface, where the area element is $d\boldsymbol{a} = -\hat{\boldsymbol{z}}dx\,dy$ (minus because the outward vector points down). This gives $\boldsymbol{f} = -\hat{\boldsymbol{z}}T_{zz} = -\sigma^2/2\epsilon_0\hat{\boldsymbol{z}}$.

   (c) The momentum flux density in direction $\boldsymbol{n}$ is given by $-\boldsymbol{n}\cdot\boldsymbol{T}$. In our case this is again $-T_{zz} = -\sigma^2/2\epsilon_0$.

   (d) This amount of momentum flux density is absorbed in each unit area of the plate, thus the force density is $\boldsymbol{f} = -\sigma^2/2\epsilon_0\hat{\boldsymbol{z}}$, as before.

2. Griffiths problem 8.9a-b (Solenoid with a ring outside, energy flux).

   Solution:

   (a) The EMF in this wire is given by $\mathcal{E} = -d\Phi/dt$, where $\Phi(t) = \pi a^2\,\mu_0 n I_s(t)$; so $\mathcal{E} = -\mu_0 n dI_s/dt$. This circuit is closed by the resistance of the wire, giving us $\mathcal{E} = I_r R$. The current in the wire is then $I_r = \frac{-1}{R}\pi a^2\,\mu_0 n\frac{dI_s}{dt}$.

   (b) Just outside the solenoid, there is an azimuthal electric field due to the changing magnetic field. We compute it from $\nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$ which leads to $\oint\boldsymbol{E}\cdot d\boldsymbol{l} = -d\Phi/dt$, from which we find $2\pi a E_\phi = -\pi a^2\,\mu_0 n dI_s/dt$.

   Meanwhile, we can find the magnetic field due to the outer ring of wire. Since the radius b of the wire is very large compared to the radius a of the solenoid, we will approximate this with the magnetic field perfectly along the axis of a loop, $\boldsymbol{B} = \hat{\boldsymbol{z}}(\mu_0 I_r/2)b^2/(b^2+z^2)^{3/2}$.

   Combine these to get the Poynting flux, $\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}$. We find

   $$\boldsymbol{S} = \frac{1}{\mu_0}\left(\frac{-a\mu_0 n}{2}\frac{dI_s}{dt}\right)\left(\frac{\mu_0 I_r}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\right)\hat{\boldsymbol{r}} . \tag{2}$$

   We integrate this energy flux over all azimuthal angles, giving a factor of $2\pi a$, and then over all values of z. This gives

   $$\frac{dE}{dt} = -\frac{\pi a^2 b^2 n\mu_0 I_r}{2}\frac{dI_s}{dt}\int_{-\infty}^{+\infty}\frac{dz}{(b^2+z^2)^{3/2}} . \tag{3}$$

This integral can be performed,

$$\int_{-\infty}^{+\infty} \frac{dz}{(b^2+z^2)^{3/2}} = \left.\frac{z}{b^2\sqrt{z^2+b^2}}\right|_{-\infty}^{+\infty} = \frac{2}{b^2}\,. \tag{4}$$

Thus we find

$$\frac{dE}{dt} = -\pi a^2 n\mu_0 I_r \frac{dI_s}{dt} = I_r^2 R\,. \tag{5}$$

3. **Solenoid filled with plasma**. Suppose we have a solenoid of radius a aligned with the $\hat{z}$ axis with n turns per unit length, and the inside of the solenoid has been filled with a low-density plasma. The plasma has a number density n_p and the charge carriers (electrons, say) each have charge q and mass m. All the charge carriers are at rest at time $t = 0$.

   (a) Suppose we turn on the current through the solenoid so that between $t = 0$ and $t = \tau$, the current increases linearly, $I(t) = I_1 \frac{t}{\tau}$. What is the magnetic field $\boldsymbol{B}$ in the solenoid? From Maxwell's equations, what is the electric field $\boldsymbol{E}$? (For this step you can ignore the influence of the plasma).

   Solution: If the current is increasing slowly (so it is in the quasistatic regime, $\tau \gg ca$), the magnetic field will be given by $\boldsymbol{B}(t) = \mu_0 n I(t)\hat{\boldsymbol{z}} = \mu_0 n I_1 t/\tau \hat{\boldsymbol{z}}$. In response to this slowly time-varying magnetic field, an azimuthal electric field will be created. From $\nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$, integrate across a disk of radius s confined to $z = const$. From the curl theorem this gives $\oint \boldsymbol{E}\cdot d\boldsymbol{l} = -d\Phi/dt$ where $\Phi = \int \boldsymbol{B}\cdot d\boldsymbol{a}$ is the flux passing through the disk. This gives us

$$2\pi s E_\phi = -\frac{d}{dt}\pi s^2 \mu_0 n I_1 \frac{t}{\tau} \tag{6}$$

$$\boldsymbol{E} = -\frac{s\mu_0 n I_1}{2\tau}\hat{\boldsymbol{\phi}}\,. \tag{7}$$

   (b) Find the energy flux and the momentum density in the electromagnetic field inside the solenoid.

   Solution: The energy flux is given by the Poynting vector,

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} \tag{8}$$

$$= \frac{1}{\mu_0}\frac{-s\mu_0 n I_1}{2\tau}\mu_0 n I_1 \frac{t}{\tau}\hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{z}} \tag{9}$$

$$\boldsymbol{S} = -\frac{\mu_0 s n^2 I_1^2 t}{2\tau^2}\hat{\boldsymbol{s}}\,. \tag{10}$$

   You can see that energy is flowing towards the axis from the edge of the solenoid. The momentum density is $\boldsymbol{\wp} = \mu_0\epsilon_0\boldsymbol{S}$ and so is proportional to the above.

   (c) From the Lorentz force on the charge carriers, solve for the motion of any given charge within the solenoid that starts out at a radius s from the axis of symmetry. (You can imagine it could be quite difficult to self-consistently solve for changing electromagnetic fields and the moving charges, since they all affect each other – which is why we ignored the influence of the plasma in item 3a).

   Solution: We have to start with velocity and acceleration given in cylindrical coordinates. If this looks unfamiliar, come ask me. We are working on a particle whose trajectory has coordinates $s(t), \phi(t), z(t)$. Then its velocity and acceleration are

$$\boldsymbol{v} = \dot{s}\hat{\boldsymbol{s}} + s\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\boldsymbol{z}} \tag{11}$$

$$\boldsymbol{a} = (\ddot{s} - s\dot{\phi}^2)\hat{\boldsymbol{s}} + (s\ddot{\phi} + 2\dot{s}\dot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\boldsymbol{z}}\,. \tag{12}$$

   Now let's compute the $\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}$. This is not too bad, since $\boldsymbol{E}$ is only in the ϕ direction, and $\boldsymbol{B}$ is only in the z direction. The result is

$$\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B} = E_\phi\hat{\boldsymbol{\phi}} - \dot{s}B_z\hat{\boldsymbol{\phi}} + s\dot{\phi}B_z\hat{\boldsymbol{s}}\,. \tag{13}$$

Now we can take the Lorentz force law, and get three equations, one from each component. The result is:

$$\ddot{z} = 0 \tag{14}$$

$$\ddot{s} - s\dot{\phi}^2 = \frac{q}{m} s\dot{\phi} B_z \tag{15}$$

$$s\ddot{\phi} + 2\dot{s}\dot{\phi} = \frac{q}{m}(E_\phi - B_z \dot{s}) \tag{16}$$

Equation (14) is trivial to solve: there is no acceleration in the z direction, so the z velocity is constant, $z(t) = z(0) + v_z(0)t$.

Now we have to be a bit more clever. The system is rotationally symmetric about the z axis, so we should think about the z component of angular momentum (over mass), $j_z = s^2\dot{\phi}$. Notice that

$$\frac{d}{dt}\left(s^2\dot{\phi}\right) = s^2\ddot{\phi} + 2s\dot{s}\dot{\phi}\,. \tag{17}$$

If we multiply Eq. (16) by s, its left hand side will be the right hand side of Eq. (17), so we get

$$\frac{d}{dt}\left(s^2\dot{\phi}\right) = \frac{q}{m}(E_\phi - B_z\dot{s})s\,. \tag{18}$$

To make more progress we need to actually see the time and s dependence of $E_\phi = -\frac{s\mu_0 n I_1}{2\tau}$ and $B_z = \mu_0 n I_1 t/\tau$. Let us define a constant $b = \mu_0 n I_1/\tau$, so $B_z = bt$, and $E_\phi = -sb/2$. Now we have

$$\frac{d}{dt}\left(s^2\dot{\phi}\right) = -\frac{bq}{m}(s^2/2 + ts\dot{s})\,. \tag{19}$$

Now the right hand side is the time derivative of $-\frac{bq}{2m}s^2 t$! This is no accident – it actually comes from Noether's theorem. So, this equation of motion gave us

$$\frac{d}{dt}\left(s^2\dot{\phi} + \frac{bq}{2m}s^2 t\right) = 0 \tag{20}$$

which means that we are differentiating some constant,

$$s^2\dot{\phi} + \frac{bq}{2m}s^2 t = \ell_0\,. \tag{21}$$

The value of this constant depends on the motion of the particle at $t = 0$; you can evaluate it, it's simply $s(0)^2\dot{\phi}(0) = j_z(0)$. Notice that $j_z = s^2\dot{\phi}$ itself is not constant, but the sum in Eq. (21) is constant.

Now we can solve for $\dot{\phi}$ in Eq. (21), and use that to eliminate it from the other equation of motion.

$$\dot{\phi} = \frac{\ell_0}{s^2} - \frac{bq}{2m}t\,. \tag{22}$$

If we had a solution for $s(t)$, then we would know the solution for $\phi(t)$ by integrating the right hand side with respect to time. But for now, we plug Eq. (22) into Eq. (15), giving an equation only in terms of s,

$$\ddot{s} - s\left(\frac{\ell_0}{s^2} - \frac{bq}{2m}t\right)^2 = \frac{q}{m}s\left(\frac{\ell_0}{s^2} - \frac{bq}{2m}t\right)bt\,. \tag{23}$$

A bit of algebra gives

$$\ddot{s} = \frac{\ell_0^2}{s^3} - \frac{b^2q^2}{4m^2}st^2\,. \tag{24}$$

Once again we can integrate the left side if we multiply everything by $2s$. This gives

$$\frac{d}{dt}\left(s^2\right) = \frac{2\ell_0^2}{s^2} - \frac{b^2q^2}{2m^2}s^2t^2\,. \tag{25}$$

But notice that this is just a first order differential equation for the variable $u = s^2$,

$$\frac{du}{dt} = \frac{2\ell_0^2}{u} - \frac{b^2q^2}{2m^2}ut^2\,. \tag{26}$$

Believe it or not, this differential equation can be solved in terms of a named special function (the *error function*, which is an integral of a Gaussian).

The general solution is tricky, but for the special case of $\ell_0 = 0$, we can find $u = s(t)^2 = Ce^{-(bqt/2m)^2}$ where $C = s(0)^2$ is a constant given by the particle's initial radius squared. This solution tells us the particle will approach the center.

Similarly, the $\dot{\phi}$ equation [Eq. (22)] can be trivially solved in the $\ell_0 = 0$ case, and the particle will simply circulate with an increasing frequency, $\phi(t) = -\frac{bq}{4m}t^2$. So, when a particle starts at rest, it will spiral inwards with an increasing frequency!

(d) From the result of item 3a, find the Maxwell stress tensor in the basis of $\hat{z}, \hat{s}, \hat{\phi}$ (i.e. you are looking for components like $T_{\phi\phi}$, T_{zs}, etc.).

Solution: It is very easy to compute E^2 and B^2,

$$E^2 = \left(\frac{s\mu_0 n I_1}{2\tau}\right)^2 \tag{27}$$

$$B^2 = \left(\mu_0 n I_1 \frac{t}{\tau}\right)^2\,. \tag{28}$$

Now examine the form of T_{ij},

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)\,. \tag{29}$$

We can write each term as a matrix in the basis (s, ϕ, z). The electric part is

$$\begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \epsilon_0 E^2\,, \tag{30}$$

and similarly the magnetic part ic

$$\begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \frac{1}{\mu_0} B^2\,. \tag{31}$$

The full Maxwell stress tensor is the sum of these two terms.

4. Griffiths problem 9.2 (Standing waves are superposed traveling waves).

Solution: Suppose we have the standing wave $f(z,t) = A\sin(kz)\cos(kvt)$. Check that it solves the wave equation:

$$\frac{\partial f}{\partial z} = kA\cos(kz)\cos(kvt) \tag{32}$$

$$\frac{\partial^2 f}{\partial z^2} = -k^2A\sin(kz)\cos(kvt) = -k^2 f \tag{33}$$

$$\frac{\partial f}{\partial t} = -kvA\sin(kz)\sin(kvt) \tag{34}$$

$$\frac{\partial^2 f}{\partial t^2} = -(kv)^2A\sin(kz)\cos(kvt) = -(kv)^2 f\,. \tag{35}$$

Thus evaluating the wave equation

$$\frac{-1}{v^2}\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}(kv)^2 f - k^2 f = 0 \quad \checkmark \tag{36}$$

Next representing this as a superposition of left and right traveling waves. All we need is the trig addition identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. You can immediately verify that

$$f(z,t) = A\sin(kz)\cos(kvt) = \frac{A}{2}\left[\sin(kz - kvt) + \sin(kz + kvt)\right] . \tag{37}$$

5. Griffiths problem 9.5 (Wave incident on a boundary where two materials meet).

Solution: We will impose two boundary conditions at the interface $z = 0$: first, $f(0^-, t) = f(0^+, t)$, and second, $\frac{\partial f}{\partial z}\big|_{0^-} = \frac{\partial f}{\partial z}\big|_{0^+}$. Our functions to the left and right are:

$$f(z,t) = \begin{cases} g_I(z - v_1 t) + h_R(z + v_1 t) & z < 0 \\ g_T(z - v_2 t) & z > 0 \,. \end{cases} \tag{38}$$

The first B.C. we impose is $f(t, 0^-) = f(t, 0^+)$. This tells us that

$$g_I(-v_1 t) + h_R(+v_1 t) = g_T(-v_2 t) \tag{39}$$

$$g_I(w) + h_R(-w) = g_T(\frac{v_2}{v_1} w) \,, \tag{40}$$

where we have defined $w \equiv v_1 t$. The second B.C. we impose is

$$\left.\frac{\partial f}{\partial z}\right|_{0^+} = \left.\frac{\partial f}{\partial z}\right|_{0^+} \tag{41}$$

$$g_I'(w) + h_R'(-w) = g_T'(\frac{v_2}{v_1} w) \,. \tag{42}$$

Let us integrate this with respect to w. The functions h_R' and g_T' will need variable substitutions to perform the integrals, giving

$$g_I(w) - h_R(-w) = \frac{v_1}{v_2} g_T(\frac{v_2}{v_1} w) + C_1 \,, \tag{43}$$

with some integration constant C_1 that depends on initial conditions. The two equations (40) and (43) form a linear system for $h_R(-w)$ and $g_T(v_2 w/v_1)$. Add the two equations and change w to $u = v_2 w/v_1$ to find

$$g_T(u) = \frac{2v_2}{v_1 + v_2} g_I(\frac{v_1}{v_2} u) + C_2 \,, \tag{44}$$

where $C_2 = -C_1 v_2/(v_1 + v_2)$. Now eliminating g_T from the system, we solve for $h_R(-w)$ (and now present it as a function of $u = -w$),

$$h_R(w) = \frac{v_1 - v_2}{v_1 + v_2} g_I(-w) + C_2 \,. \tag{45}$$

6. Griffiths problem 9.8 (Circularly polarized wave).

Solution: We are working with the wave

$$\tilde{\boldsymbol{f}} = \tilde{A} e^{i(kz - \omega t)} \left(e^{i\delta_v} \hat{\boldsymbol{x}} + e^{i\delta_h} \hat{\boldsymbol{y}}\right) \tag{46}$$

where we have set the $\hat{\boldsymbol{x}}$ direction to be vertical, and we are using $\delta_v = 0, \delta_h = \pi/2$, so

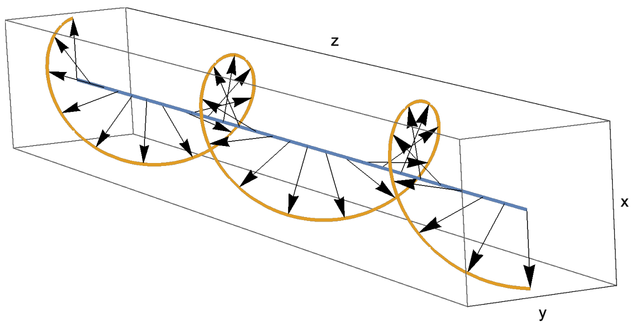
$$\tilde{\boldsymbol{f}} = \tilde{A} e^{i(kz - \omega t)} \left(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}}\right) . \tag{47}$$

(a) The physical string motion is given by $\boldsymbol{f} = \text{Re}[\hat{\boldsymbol{f}}]$. Without loss of generality, let's choose $\tilde{A} = A$ to be real, and take $z = 0$. The string motion will be

$$\boldsymbol{f} = A\,\text{Re}[e^{-i\omega t}\,(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})] \tag{48}$$
$$= A\,[\cos(-\omega t)\hat{\boldsymbol{x}} + \cos(\pi/2 - \omega t)\hat{\boldsymbol{y}}] = A\,[\cos(\omega t)\hat{\boldsymbol{x}} + \sin(\omega t)\hat{\boldsymbol{y}}]\ . \tag{49}$$

This is parametrically describing a point moving around the circle of radius A in the $x - y$ plane. At time $t = 0$, it is on the positive x axis. At time $\pi/2\omega$, it is on the positive y axis. Thus if you are at positive z and looking toward the origin, you will see the point circles **counter-clockwise**. To make it circle clockwise, set $\delta_h = -\pi/2$.

(b)



(c) Just hold the end while moving your wrist or arm periodically in a steady circle.