UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism II (Phys. 402) — Prof. Leo C. Stein — Spring 2019

Problem Set 2 — SOLUTIONS

Due: Thursday, Feb. 21, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. Griffiths problem 7.30 (Mutual inductance between two tiny wire loops).

   Solution:

   (a) Since mutual inductance is reciprocal, we can take either loop as having a sustained current, and compute the current induced in the other. Put loop 1 at the origin. If loop 1 has current I_1, it generates approximately $\boldsymbol{B}_1 = \frac{\mu_0}{4\pi}\frac{1}{r^3}I_1[3(\boldsymbol{a}_1 \cdot \hat{\boldsymbol{\imath}})\hat{\boldsymbol{\imath}} - \boldsymbol{a}_1]$. Then the flux through loop 2 is $\Phi_2 = \boldsymbol{B}(\boldsymbol{r} = \boldsymbol{r}_2)\cdot\boldsymbol{a}_2 = \frac{\mu_0}{4\pi}\frac{1}{r^3}I_1[3(\boldsymbol{a}_1\cdot\hat{\boldsymbol{\imath}})(\hat{\boldsymbol{\imath}}\cdot\boldsymbol{a_2})-\boldsymbol{a}_1\cdot\boldsymbol{a}_2] = MI_1$. Thus $M = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\boldsymbol{a}_1\cdot\hat{\boldsymbol{\imath}})(\hat{\boldsymbol{\imath}}\cdot\boldsymbol{a_2})-\boldsymbol{a}_1\cdot\boldsymbol{a}_2]$ which is clearly symmetric under the exchange $1 \leftrightarrow 2$.

   (b) Due to the changing current $I_2(t) \neq$ const., there will be an EMF in loop one, $\mathcal{E}_1 = -M\frac{d}{dt}I_2$. If a current controller on loop 1 is keeping its current at a constant I_1 then the work it's doing per unit time is $\frac{dW}{dt} = -\mathcal{E}_1 I_1 = MI_1\frac{d}{dt}I_2$. Integrate this over the time it takes to change the current in loop 2, but the whole thing is just a constant times the total derivative $\frac{d}{dt}I_2$. Thus the work that the current controller performs is $W = MI_1I_2 = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\boldsymbol{m}_1 \cdot \hat{\boldsymbol{\imath}})(\hat{\boldsymbol{\imath}} \cdot \boldsymbol{m_2}) - \boldsymbol{m}_1 \cdot \boldsymbol{m}_2]$ where $\boldsymbol{m}_1 = \boldsymbol{a}_1 I_1$ and similarly for loop 2. This is the same as Eq. (6.35) except for a minus sign. That equation only accounted for the energy in the fields, whereas here we have the energy in the source currents as well. The fact that the sum of (fields+sources) is the same as (-fields) is extremely common in linear theories (such as electromagnetism) and can be explained more deeply from a Lagrangian or Hamiltonian point of view.

2. Much of the interstellar medium (ISM) is very low number density, typically $n \approx 1\text{atom/cm}^3$ (you can assume it is entirely Hydrogen). There are magnetic fields permeating the galaxy with strengths like $B \approx 10\mu G$ (microGauss).

   (a) What is a typical energy density of the magnetic field in the ISM?

   Solution: Energy density in the magnetic field is $u_B = \frac{1}{2\mu_0}B^2 \approx 4 \times 10^{-13} J/m^3$ (in SI units) or $\approx 4 \times 10^{-12} erg/cm^3$ (in cgs units).

   (b) Suppose there is *equipartition* of energy between magnetic energy density and thermal energy density (which of course depends on density and temperature). What is a typical temperature of the ISM?

   Solution: Assuming equipartition, we would expect $u_B \approx u_{th}$, where the energy density in a thermal gas is $u_{th} = n\langle E\rangle$ where $\langle E\rangle$ is the average energy per particle and n is the number density. The average energy per degree of freedom is $\frac{1}{2}k_BT$ where k_B is Boltzmann's constant. Most of the gas in the ISM is a single Hydrogen atom, so it only has translation degrees of freedom (no rotational or internal), so $\langle E\rangle = \frac{3}{2}k_BT$. Therefore we expect $u_B = \frac{3}{2}nk_BT$ and can solve for temperature $T = \frac{B^2}{3\mu_0 nk_B} \approx 2 \times 10^4 K$. This is not too far off from temperatures of the "warm neutral medium."

3. Griffiths 4^{th} edition problem 7.49 (getting $\boldsymbol{E}$ in terms of the vector potential).

   Solution:

(a) (In the 3rd edition): In magnetostatic we have $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$. Biot-Savart says that a magnetic field that solves this is $\boldsymbol{B} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d^3 r'$. In parallel, the equations for induced electric fields are $\nabla \cdot \boldsymbol{E} = 0$ (no charges, just induced fields) and $\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t} \boldsymbol{B}$. So we can take Biot-Savart and replace $\boldsymbol{B}$ with $\boldsymbol{E}$ on the left-hand side if we replace $\mu_0 \boldsymbol{J}$ with $-\frac{\partial \boldsymbol{B}}{\partial t}$on the right-hand side. This gives the desired equation,

$$\boldsymbol{E} = \frac{-1}{4\pi} \frac{\partial}{\partial t} \int \frac{\boldsymbol{B}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d^3 r' \,. \tag{1}$$

(b) From the earlier problem, $\boldsymbol{A}$ depends on $\boldsymbol{B}$ in the same way that $\boldsymbol{B}$ depends on $\mu_0 \boldsymbol{J}$. So replacing in Biot-Savart gives

$$\boldsymbol{A} = \frac{1}{4\pi} \int \frac{\boldsymbol{B}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d^3 r' \,. \tag{2}$$

Comparing Eqs. (1) and (2) we see that $\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t$.

(c) Because electrodynamics is linear, we can superimpose the following two results. First, just due to stationary charge density, which is not changing, we get an electric field that vanishes inside the shell, and outside is given by $\boldsymbol{E}_{Coul.} = \hat{\boldsymbol{r}} \frac{Q}{4\pi\epsilon_0 r^2}$, where $Q = 4\pi R^2$, or $\boldsymbol{E}_{Coul.} = \hat{\boldsymbol{r}} \frac{\sigma R^2}{\epsilon_0 r^2}$. Second, from the currents which slowly change with time. We can take the result for $\boldsymbol{A}_{Farad.}$ from Ex. 5.11, as long as the frequency $\omega(t)$ changes very slowly in time (i.e. $d\omega/dt << \omega^2$). That result was

$$\boldsymbol{A}_{Farad.} = \begin{cases} \frac{\mu_0 R \omega \sigma}{3} r \sin\theta \hat{\boldsymbol{\phi}} \,, & (r < R), \\ \frac{\mu_0 R^4 \omega \sigma}{3} \frac{\sin\theta}{r^2} \hat{\boldsymbol{\phi}} \,, & (r > R). \end{cases} \tag{3}$$

Now we consider $\omega = \omega(t)$ to be a function of time, and use $\boldsymbol{E}_{Farad.} = -\partial \boldsymbol{A}_{Farad.}/\partial t$ to get the contribution from the current. Finally, the total electric field is the sum $\boldsymbol{E} = \boldsymbol{E}_{Coul.} + \boldsymbol{E}_{Farad.}$,

$$\boldsymbol{E} = \begin{cases} \frac{\mu_0 R \dot{\omega} \sigma}{3} r \sin\theta \hat{\boldsymbol{\phi}} \,, & (r < R), \\ \frac{\sigma R^2}{\epsilon_0 r^2} \hat{\boldsymbol{r}} + \frac{\mu_0 R^4 \dot{\omega} \sigma}{3} \frac{\sin\theta}{r^2} \hat{\boldsymbol{\phi}} \,, & (r > R). \end{cases} \tag{4}$$

4. Griffiths problem 7.53 (ratio of EMFs in a transformer).

Solution: Consider an individual turn of the winding of either the primary or secondary coil. Suppose the amount of flux through this one turn is Φ. Then the amount of flux through the primary and secondary is (respectively) $\Phi_1 = N_1 \Phi$ and $\Phi_2 = N_2 \Phi$. Thus the respective EMFs are $\mathcal{E}_1 = -\frac{d}{dt}\Phi_1 = -N_1 \frac{d}{dt}\Phi$ and similarly $\mathcal{E}_2 = -N_2 \frac{d}{dt}\Phi$. Thus we immediately find $\mathcal{E}_1/\mathcal{E}_2 = N_1/N_2$.

5. **A highly conducting, magnetized plasma**. Consider a plasma with a conductivity[1] σ, charge density ρ (that varies throughout the plasma), and where at each point the particles are moving with velocity $\boldsymbol{v}$ (that also varies from place to place). Ohm's law says that the current density is

$$\boldsymbol{J} = \sigma \boldsymbol{f} = \sigma (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \,. \tag{5}$$

(a) Suppose the conductivity σ is taken to infinity, while the current density $\boldsymbol{J} = \rho \boldsymbol{v}$ remains finite. What relationship does this imply between the electromagnetic fields?

Solution: For $\boldsymbol{J}$ to remain finite while $\sigma \to \infty$, we need $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \to 0$ at the same speed as $1/\sigma \to 0$. Thus we end up with

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} \,. \tag{6}$$

(b) From the previous answer, what do you know about $\boldsymbol{E} \cdot \boldsymbol{B}$?

Solution: Since $\boldsymbol{E}$ is the cross product of *something* with $\boldsymbol{B}$, the electric field must be perpendicular to the magnetic field, so

$$\boldsymbol{E} \cdot \boldsymbol{B} = 0 \,. \tag{7}$$

[1] We will not refer to the resistivity, $1/\sigma$, which is sometimes denoted ρ. Instead we reserve the symbol ρ for the charge density.

(c) From these two results, you should be able to determine $\boldsymbol{v}_\perp$, the part of the plasma velocity that is perpendicular to the magnetic field [Hint: decompose $\boldsymbol{v}$ into pieces that are parallel and perpendicular to $\boldsymbol{B}$; then you might want to cross product the result of part 5a with something else].

Solution: We write $\boldsymbol{v} = v_\parallel \hat{\boldsymbol{B}} + \boldsymbol{v}_\perp$ where $\boldsymbol{v}_\perp \cdot \boldsymbol{B} = 0$ and $\hat{\boldsymbol{B}} = \boldsymbol{B}/|B|$. Then in Eq. (6), only $\boldsymbol{v}_\perp$ contributes to the cross product, $\boldsymbol{E} = -\boldsymbol{v}_\perp \times \boldsymbol{B}$. This equation tells us that $\boldsymbol{v}_\perp$ makes a right angle with both $\boldsymbol{B}$ and $\boldsymbol{E}$, so it is proportional to $\boldsymbol{E} \times \boldsymbol{B}$ with some coefficient. You can plug in this ansatz and solve for the coefficient. Alternatively, you can take the cross product of the whole equation with $\boldsymbol{B}$,

$$\boldsymbol{B} \times \boldsymbol{E} = -\boldsymbol{B} \times (\boldsymbol{v}_\perp \times \boldsymbol{B}) \tag{8}$$

$$= -[\boldsymbol{v}_\perp(\boldsymbol{B} \cdot \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{v}_\perp \cdot \boldsymbol{B})] = -B^2 \boldsymbol{v}_\perp \,, \tag{9}$$

since $\boldsymbol{v}_\perp$ is perpendicular to $\boldsymbol{B}$. Thus we have $\boldsymbol{v}_\perp = \boldsymbol{E} \times \boldsymbol{B}/B^2$.

(d) Take the time derivative of the result of part 5b. Plug in for the time derivatives of the electromagnetic fields using Maxwell's equations. You should now be able to solve for $\boldsymbol{v} \cdot \boldsymbol{B}$, allowing you to find the parallel component $v_\parallel$.

Solution: The partial time derivative of Eq. (7) is

$$0 = \boldsymbol{B} \cdot \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \,. \tag{10}$$

For the time derivatives of the fields we use Maxwell's equations, so we get

$$0 = \boldsymbol{B} \cdot \frac{1}{\mu_0 \epsilon_0} [\nabla \times \boldsymbol{B} - \mu_0 \boldsymbol{J}] - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{E}) \,. \tag{11}$$

Now recall that $\boldsymbol{J} = \rho \boldsymbol{v} = \rho(v_\parallel \hat{\boldsymbol{B}} + \boldsymbol{v}_\perp)$. Once you insert this you will be able to solve for $v_\parallel$,

$$\rho v_\parallel \boldsymbol{B} \cdot \hat{\boldsymbol{B}} = \frac{1}{\mu_0} \boldsymbol{B} \cdot (\nabla \times \boldsymbol{B}) - \epsilon_0 \boldsymbol{E} \cdot (\nabla \times \boldsymbol{E}) \,, \tag{12}$$

$$v_\parallel = \frac{1}{\rho |B|} \left[\frac{1}{\mu_0} \boldsymbol{B} \cdot (\nabla \times \boldsymbol{B}) - \epsilon_0 \boldsymbol{E} \cdot (\nabla \times \boldsymbol{E}) \right] \,. \tag{13}$$

We can even replace ρ with $\epsilon_0 \nabla \cdot \boldsymbol{E}$ in the denominator.

Thus in the infinite conductivity limit, ρ and $\boldsymbol{v}$ can be solved for in terms of the electromagnetic fields, and eliminated from the equations. The resulting system of equations and physical systems are called *Force-free electrodynamics*.

If you were successful in all these steps, then (in this highly conductivity regime) you can eliminate the matter sources from Maxwell's equations. This regime is relevant in many astrophysical plasmas that are very low density.

6. Griffiths problem 7.59 (proving Alfven's theorem).

Solution:

(a) As in the previous problem, for $\boldsymbol{J}$ to remain finite while $\sigma \to \infty$, we need $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \to 0$ at the same speed as $1/\sigma \to 0$. So we start with $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$. Take the curl of both sides, $\nabla \times \boldsymbol{E} = -\nabla \times (\boldsymbol{v} \times \boldsymbol{B})$. But the left hand side can be replaced, by Faraday's law, with $-\partial \boldsymbol{B}/\partial t$, so we have arrived at the desired result, $\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$.

(b) From $\nabla \cdot \boldsymbol{B} = 0$, integrating over any volume and using the divergence theorem, $\oint \boldsymbol{B} \cdot d\boldsymbol{a} = 0$ over any closed surface. The three surfaces $\mathcal{S}$, $\mathcal{R}$, and $\mathcal{S}'$ together make a closed surface, but the orientation of (direction of area element) is not consistent for the three surfaces. To make the orientation consistent we need a sign difference between $\int_{\mathcal{S}} \boldsymbol{B}(t+dt)\cdot d\boldsymbol{a}$ and $\int_{\mathcal{S}'} \boldsymbol{B}(t+dt)\cdot d\boldsymbol{a}$. That gives the desired equation

$$\int_{\mathcal{S}'} \boldsymbol{B}(t+dt)\cdot d\boldsymbol{a} + \int_{\mathcal{R}} \boldsymbol{B}(t+dt)\cdot d\boldsymbol{a} = \int_{\mathcal{S}} \boldsymbol{B}(t+dt)\cdot d\boldsymbol{a}\,. \tag{14}$$

Now you can replace $\int_{\mathcal{S}'} \boldsymbol{B}(t+dt)\cdot d\boldsymbol{a}$ in the equation for the flux change,

$$d\Phi = \int_{\mathcal{S}} \boldsymbol{B}(t+dt)\cdot d\boldsymbol{a} - \int_{\mathcal{S}} \boldsymbol{B}(t)\cdot d\boldsymbol{a} - \int_{\mathcal{R}} \boldsymbol{B}(t+dt)\cdot d\boldsymbol{a}\,. \tag{15}$$

The first two terms are over the same surface, and we can use Taylor's theorem to approximate $\boldsymbol{B}(t+dt) = \boldsymbol{B}(t) + dt\ \partial\boldsymbol{B}/\partial t$, giving

$$d\Phi = dt\int_{\mathcal{S}} \frac{\partial \boldsymbol{B}}{\partial t}\cdot d\boldsymbol{a} - \int_{\mathcal{R}} \boldsymbol{B}(t+dt)\cdot d\boldsymbol{a}\,. \tag{16}$$

For the last integral, the ribbon is a Cartesian product of the loop $\mathcal{P} \equiv \partial\mathcal{S}$ (the boundary of $\mathcal{S}$) times a short segment generated by $\boldsymbol{v}dt$. Thus we can parameterize each area element along $\mathcal{R}$ with $d\boldsymbol{a} = d\boldsymbol{l}\times\boldsymbol{v}dt$ (this has the same orientation as $\mathcal{S}$ if $\mathcal{P}$ is traversed counter-clockwise as it is drawn in the book). Then we permute the triple product $\boldsymbol{B}\cdot(d\boldsymbol{l}\times\boldsymbol{v})dt = (\boldsymbol{v}\times\boldsymbol{B})\cdot d\boldsymbol{l}dt$. Next we use the curl theorem, backwards, to turn this last integral into one over $\mathcal{S}$,

$$d\Phi = dt\left\{\int_{\mathcal{S}} \frac{\partial \boldsymbol{B}}{\partial t}\cdot d\boldsymbol{a} - \int_{\mathcal{S}} \nabla\times(\boldsymbol{v}\times\boldsymbol{B})d\boldsymbol{a}\right\}\,. \tag{17}$$

But if we combine integrands, the total integrand is $\partial\boldsymbol{B}/\partial t - \nabla\times(\boldsymbol{v}\times\boldsymbol{B})$, which from item 6a vanishes identically.