UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism I (Phys. 401) — Prof. Leo C. Stein — Fall 2019

Problem Set 10

Due: Friday, Dec. 6, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. Suppose we take a flat disk of radius R, and uniformly distribute a charge Q on the disk, so it has surface charge density $\sigma = Q/(4\pi R^2)$. Now suppose we spin the disk about its symmetry axis, at an angular velocity ω. What is the resulting surface current distribution $\boldsymbol{K}$? What is the magnetic dipole moment $\boldsymbol{m}$ of this current distribution?

2. We saw that the field of an ideal magnetic dipole m aligned with the z axis was

$$\boldsymbol{B} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}})\,. \tag{1}$$

Show that this can be written in an arbitrary coordinate system as

$$\boldsymbol{B} = \frac{\mu_0}{4\pi r^3}[3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}]\,. \tag{2}$$

3. Previously we saw the first few terms of the multipole expansion of $\boldsymbol{A}$ created by a line current $\boldsymbol{I}$,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left\{\frac{\boldsymbol{0}}{r} + \frac{\boldsymbol{m}\times\hat{\boldsymbol{r}}}{r^2} + \ldots\right\}\,. \tag{3}$$

Let's generalize this to volume currents.

(a) Starting from the integral solution of $\boldsymbol{A}(\boldsymbol{r})$ in terms of $\boldsymbol{J}(\boldsymbol{r}')$, develop the multipole expansion. Your result should be a power series of the form

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\boldsymbol{X}_n\,, \tag{4}$$

where $\boldsymbol{X}_n$ is some integral involving $\boldsymbol{J}$ and Legendre polynomials.

(b) Show that the term $\boldsymbol{X}_0$ must vanish.

(c) Recall the line current definitions $\boldsymbol{m} \equiv I\boldsymbol{a}$, where $\boldsymbol{a} \equiv \int_{\mathcal{A}} d^2\boldsymbol{a}$. Show that this directed area element also equals

$$\boldsymbol{a} = \frac{1}{2}\oint_{\partial\mathcal{A}}\boldsymbol{r}\times d\boldsymbol{l}\,. \tag{5}$$

This involves a little bit of the geometrical meaning of the cross product.

(d) Starting from $\boldsymbol{m} \equiv I\boldsymbol{a}$, using the previous result and now promoting to a volume current $\boldsymbol{J}d^3\mathrm{Vol}$, show that the dipole moment is

$$\boldsymbol{m} = \frac{1}{2}\int \boldsymbol{r}\times\boldsymbol{J}\,d^3\mathrm{Vol}\,. \tag{6}$$

4. In electrostatics, for the force on an electric dipole due to an external electric field, we could write either $\boldsymbol{F} = \nabla(\boldsymbol{p}\cdot\boldsymbol{E})$ or $\boldsymbol{F} = (\boldsymbol{p}\cdot\nabla)\boldsymbol{E}$. (i) Prove this. However in magnetostatics, the force on a magnetic dipole in an external magnetic field can only be written as $\boldsymbol{F} = \nabla(\boldsymbol{m}\cdot\boldsymbol{B})$. (ii) Explain why.

5. (a) Suppose we bring a magnetic dipole $\boldsymbol{m}$ from infinity into an external magnetic field $\boldsymbol{B}(\boldsymbol{r})$. As you bring it in, the magnitude m must remain constant, though you may change the orientation. Show that associated energy is

$$U = -\boldsymbol{m} \cdot \boldsymbol{B} \, . \tag{7}$$

(b) Show that the interaction energy between two magnetic dipoles $\boldsymbol{m}_1$, $\boldsymbol{m}_2$, separated by $\boldsymbol{r}$, is

$$U = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[\boldsymbol{m}_1 \cdot \boldsymbol{m}_2 - 3(\boldsymbol{m}_1 \cdot \hat{\boldsymbol{r}})(\boldsymbol{m}_2 \cdot \hat{\boldsymbol{r}}) \right] \, . \tag{8}$$

(c) Suppose we fix $\boldsymbol{r}$, so only the dipole orientations can change. For simplicity, let's say the dipoles are positioned along the z axis, and both oriented within the $x - z$ plane. Express the energy U in terms of the angles θ_1 and θ_2, measured from the z axis. What is the condition on U, in terms of θ_1 and θ_2, for the orientation of both dipoles to be stable?

(d) Identify the most stable configuration (i.e. angles θ_1 and θ_2). You may have to compare a few different orientations.