**UNIVERSITY OF MISSISSIPPI**
Department of Physics and Astronomy
Electromagnetism I (Phys. 401) — Prof. Leo C. Stein — Fall 2019

**Problem Set 7**

**Due**: Monday, Oct. 28, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. **Basis decompositions**. This is practice for decomposing functions into some of the more common bases: series of Legendre polynomials $P_n(x)$, (complex) Fourier series $e^{inx}$, and spherical harmonics $Y_l^m(\theta,\phi)$. You can perform the decompositions by identifying coefficients, or the more systematic approaching of using orthogonality & completeness of basis functions. The orthogonality relations are:

$$\int_0^{2\pi} e^{-inx} e^{imx} dx = 2\pi\delta_{n,m} \tag{1}$$

$$\int_{-1}^{+1} P_n(x) P_m(x) dx = \frac{2}{2n+1}\delta_{n,m} \tag{2}$$

$$\int_0^{2\pi}\int_0^{\pi} \overline{Y_l^m(\theta,\phi)} Y_{l'}^{m'}(\theta,\phi) \sin\theta \, d\theta \, d\phi = \delta_{l,l'}\delta_{m,m'} \tag{3}$$

where $n, m, l, l', m'$ are all integers.

   (a) Decompose $p(u) = 3u^3 - 4u^2 + u$ as a series of Legendre polynomials $P_n(u)$
   (b) Decompose $q(\theta) = \cos^5(\theta)$ as a series of Legendre polynomials $P_n(u)$
   (c) Decompose $r(z) = 1 + \sin^4(z)$ as a complex Fourier series
   (d) Decompose $w(\theta,\phi) = 4\cos\theta\sin^2\theta\sin(2\phi)$ into spherical harmonics $Y_l^m(\theta,\phi)$

2. **Potential inside a cube**. Your friend Brittany has constructed a cubical box with all sides of length $L$, going from the origin to $x = L$, $y = L$, and $z = L$. Five sides are held at ground, potential $V = 0$. The side in the $x - y$ plane with $z = 0$ is insulated from the other sides and has a constant potential $V = V_0$. What is the potential $V(x, y, z)$ (which depends on all three coordinates) everywhere inside the box?

3. **Oppositely charged hemispheres**. Claire has prepared two insulating hemispherical shells of radius $R$, centered on the origin. The top hemisphere (with $z > 0$ or $0 \le \theta < \pi/2$) has uniform surface charge density $+\sigma_0$, while the opposite hemisphere ($z < 0$ or $\pi/2 < \theta \le \pi$) has the opposite uniform surface charge density, $-\sigma_0$. Find the first three *nonvanishing* terms of the Legendre expansion of the potential $V(\boldsymbol{r})$ both inside and outside the shell of charge. Remember that the coefficients $A_l, B_l$ are different inside and out.

4. **Continuing the multipole expansion**. We've already seen in lecture that the multipole expansion can be expressed either in terms of a Legendre expansion,

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int (r')^{\ell} P_\ell(\cos\gamma)\rho(\boldsymbol{r}')\, d^3\mathrm{Vol}' \,, \tag{4}$$

where $\cos\gamma = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}'$; but we've also seen that we can write the first three term as

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{Q}{r} + \frac{p_i \hat{r}^i}{r^2} + \frac{1}{2}\frac{Q_{ij}\hat{r}^i\hat{r}^j}{r^3} + \dots \right\} , \tag{5}$$

where, by expanding the Legendre polynomials, we've defined the scalar $Q$, vector $p_i$, and a rank-2 tensor $Q_{ij}$,

$$Q = \int \rho(\boldsymbol{r}')\, d^3\mathrm{Vol}' \tag{6}$$

$$p_i = \int r'_i \rho(\boldsymbol{r}')\, d^3\mathrm{Vol}' \tag{7}$$

$$Q_{ij} = \int [3r'_i r'_j - (r')^2 \delta_{ij}]\rho(\boldsymbol{r}')\, d^3\mathrm{Vol}' \,. \tag{8}$$

(a) Find the next term in the series of Eq. (5) by defining an appropriate rank-3 tensor $O_{ijk}$. Give both your definition for $O_{ijk}$ and how this tensor appears in the expansion of $V(\boldsymbol{r})$ (this way you can get credit if e.g. you include a factor of 2 in one place but compensate with a factor of $\frac{1}{2}$ in another place).

(b) Suppose we arrange charges at the four corners of a square in rectangular coordinates at $(\pm d, \pm d, 0)$. The charges in the $++$ and $--$ positions are $+q$, while the charges in the $+-$ and $-+$ positions are $-q$. For this charge configuration, evaluate all the components of these tensors: $Q, p_i, Q_{ij}$. Evaluate the components of $O_{ijk}$ if you have some free time on your hands.