UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Electromagnetism I (Phys. 401) — Prof. Leo C. Stein — Fall 2019

Problem Set 2

Due: Thursday, Sept. 12, 2019, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work. All book problem numbers refer to the third edition of Griffiths, unless otherwise noted. I know we don't all have the same edition, so I also briefly describe the topic of the problem.

1. Let's define the vector field $\boldsymbol{V} = xy\hat{\boldsymbol{x}} - \frac{3}{2}y^2\hat{\boldsymbol{y}}$. Evaluate the line integral

$$I = \int_{\mathcal{P}} \boldsymbol{V} \cdot d\boldsymbol{l}\,, \tag{1}$$

where the path $\mathcal{P}$ starts at $(x, y) = (0, 0)$ and goes to $(1, 2)$ along the path parameterized by $\boldsymbol{\gamma}(\lambda) = (\lambda, 2\lambda^3)$.

2. **Checking the divergence theorem**. Consider the vector field

$$\boldsymbol{A} = xy^2\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + xyz\hat{\boldsymbol{z}}\,. \tag{2}$$

Consider a rectangular prism, aligned with the xyz axes, of length 1 in the x direction, width 1 in the y direction, and some arbitrary height h in the z direction. Let one corner of this rectangular prism sit at $(0, 0, 0)$ and the opposite corner be at $(1, 1, h)$. We will call the interior $\mathcal{V}$ and the surface $\partial\mathcal{V} = \mathcal{S}$. Recall that a surface has an orientation; we take the orientation of the 6 rectangular faces to be pointing out.

   (a) Compute the six surface integrals over the 6 oriented faces and add them up to find the total surface integral

$$I = \oint_{\mathcal{S}} \boldsymbol{A} \cdot d\boldsymbol{a}\,. \tag{3}$$

   (b) Use the divergence theorem to turn the above integral into a volume integral, and evaluate the volume integral to verify that the two approaches give the same result.

3. **Various surfaces**.

   (a) Let's define a parametric 2-dimensional surface in 3-dimensional space with the functions

$$\boldsymbol{\sigma}(u, v) \equiv (v\cos u, v\sin u, v)\,, \tag{4}$$

   where u, v are two parameters along the surface. Find the oriented differential area element $d\boldsymbol{a}$ at some point with parameters (u, v). Describe in words the shape of this surface.

   (b) Besides a parametric definition (like above) and an implicit definition (like a set of points satisfying some equation $f(x, y, z) = 0$), we can also define a surface via the *graph* of some function $z = g(x, y)$.

      i. What is a function $f(x, y, z)$ such that the points $f(x, y, z) = 0$ are the same as the set of points on the graph $z = g(x, y)$? Find a normal to surface $f(x, y, z) = 0$ in terms of $g(x, y)$.

      ii. Give a parametric form of the same surface, i.e. find a $\boldsymbol{\sigma}(u, v)$ which returns the same set of points as the graph $z = g(x, y)$. Find the oriented differential area element $d\boldsymbol{a}$ to this parametric surface. Check that this area element points in the same direction as the normal you found above.