UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 8

Due: Friday, Nov. 16, 2018, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **A slowly-changing quartic oscillator**. In lecture, we discussed the example of treating a quartic potential as a perturbation to a quadratic one. The example Hamiltonian was

$$H = H_0 + \epsilon H_1 \,, \qquad H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 q^2 \,, \qquad H_1 = \frac{1}{4} m q^4 \,. \tag{1}$$

Recall that the SHO (given by H_0) can be put into action-angle form via the transformation

$$q = \sqrt{\frac{2J_0}{m\omega_0}} \sin\phi_0 \,, \qquad p = \sqrt{2J_0 m\omega_0} \cos\phi_0 \,. \tag{2}$$

(a) Solve for $\phi_0(p, q)$ and $J_0(p, q)$ and show that these two are a canonically conjugate pair.

(b) Show that these are action-angle variables by writing $H_0(\phi_0, J_0)$. How do you know that this is an action-angle form of H_0?

(c) What is the perturbed system H in terms of the old action-angle variables, $H(\phi_0, J_0)$?

Now recall that for the perturbed system H, we could find canonical transformation from (ϕ_0, J_0) to new action-angle variables (ϕ, J). We did this with the type-2 canonical transformation.

(d) How does some type-2 generating function $F_2(\phi_0, J)$ determine the relationship between the old variables (ϕ_0, J_0) and new variables (ϕ, J)? That is, what do the two derivatives $\partial F_2/\partial\phi_0$ and $\partial F_2/\partial J$ yield?

Specifically, we had the near-identity canonical transformation

$$F_2(\phi_0, J) = \phi_0 J + \epsilon \frac{1}{m\omega_0^2} \frac{J^2}{8\omega_0} \left(2\sin^2\phi_0 + 3\right) \sin\phi_0 \cos\phi_0 \,. \tag{3}$$

(e) What is the relationship between (ϕ_0, J_0) and (ϕ, J)?

Now suppose that ϵ is a time-varying parameter $\epsilon(t)$, which varies on timescales that are very long compared to the oscillation frequency.

(f) What quantity is adiabatically invariant?

(g) Write the adiabatic invariant in terms of the original phase space variables (q, p) using the transformation given in Eq. (2) [Hint 1: In the $\mathcal{O}(\epsilon)$ pieces of the relationship given in 1e, it is consistent to replace J_0 with J or vice versa, which only incurs an error of $\mathcal{O}(\epsilon^2)$. Hint 2: using Eq. (2) to substitute for $\sin\phi_0$ and $\cos\phi_0$ is easier than plugging in some multi-valued function like arctan, as this avoids the need to identify which branch of the function you need]

Suppose that at time $t = 0$, $\epsilon(0) = 0$, and there was some maximum oscillation amplitude $q_{\max}$ (at which point the momentum p vanished).

(h) At any time t (or value of ϵ), find an equation that relates $q_{\max}$ (the max displacement, when $p = 0$) and the adiabatic invariant from the previous part.

(i) What is the explicit dependence $q_{\max,0}(J_0)$ when $\epsilon = 0$?

Supposing that the *change* in the max displacement is small, you can write the max displacement as $q_{\max} = q_{\max,0} + \epsilon\delta q_{\max}$.

(j) Plugging this approximation into the result from 1h, find an equation for $\delta q_{\max}$, in terms of the original amplitude $q_{\max,0}$.

2. **Cubic correction to the SHO**. Let us now consider a cubic correction to the SHO, by taking the same H_0 as above, but now taking the perturbation

$$H_1 = \frac{1}{3} m q^3 \,. \tag{4}$$

(a) What is the Hamiltonian $H = H_0 + \epsilon H_1$ in terms of the (old) AA vars (ϕ_0, J_0) given previously?

(b) What are the equations of motion for ϕ_0 and J_0?

Recall that angle-averaging of any quantity f is defined as

$$\langle f(\phi_0, J_0)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\phi_0, J_0) d\phi_0 \,. \tag{5}$$

(c) Average the right-hand-sides of the Hamilton's equations for ϕ_0 and J_0 over a single period of the ϕ_0 motion. In other words, compute $\langle\dot{\phi}_0\rangle$ and $\langle\dot{J}_0\rangle$.

(d) Also compute the average of the perturbation to the Hamiltonian, $\langle H_1\rangle$

(e) Comment on the system's secular behavior.

Now recall that if we want to find a type-2 near-identity generating function to put this system in AA form, we need to compute

$$F_2(\phi_0, J) = \phi_0 J + \epsilon \int^{\phi_0} \frac{\langle H_1\rangle - H_1(\phi_0', J)}{\omega_0(J)} d\phi_0' \tag{6}$$

(f) Compute the integral above, thus finding the type-2 generating function we need.

(g) With this generating function, find the relationship between (ϕ_0, J_0) and (ϕ, J).

(h) For good measure: what is a different expression for J in terms of an integral in the original (q, p) phase space variables?