UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 8 — SOLUTIONS

Due: Friday, Nov. 16, 2018, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **A slowly-changing quartic oscillator**. In lecture, we discussed the example of treating a quartic potential as a perturbation to a quadratic one. The example Hamiltonian was

$$H = H_0 + \epsilon H_1\,, \qquad H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2\,, \qquad H_1 = \frac{1}{4}mq^4\,. \tag{1}$$

Recall that the SHO (given by H_0) can be put into action-angle form via the transformation

$$q = \sqrt{\frac{2J_0}{m\omega_0}}\sin\phi_0\,, \qquad p = \sqrt{2J_0 m\omega_0}\cos\phi_0\,. \tag{2}$$

(a) Solve for $\phi_0(p,q)$ and $J_0(p,q)$ and show that these two are a canonically conjugate pair.
Solution: If we take the ratio q/p, the factor of $\sqrt{J_0}$ will cancel and we can solve for $\tan\phi_0$,

$$\tan\phi_0 = m\omega_0\frac{q}{p} \qquad \Longleftrightarrow \qquad \phi_0 = \arctan\left(m\omega_0\frac{q}{p}\right) \tag{3}$$

Meanwhile, from the identity $\sin^2\phi_0 + \cos^2\phi_0 = 1$, we can get an equation without ϕ_0, and get

$$J_0 = \frac{1}{2}\left(\frac{p^2}{m\omega_0} + m\omega_0 q^2\right)\,. \tag{4}$$

Now to show that they are a canonically conjugate pair, we need to compute the Poisson bracket $\{\phi_0, J_0\}$. After taking the derivatives, the algebra indeed shows that

$$\{\phi_0, J_0\} = \frac{\partial\phi_0}{\partial q}\frac{\partial J_0}{\partial p} - \frac{\partial\phi_0}{\partial p}\frac{\partial J_0}{\partial q} = 1\,. \tag{5}$$

(b) Show that these are action-angle variables by writing $H_0(\phi_0, J_0)$. How do you know that this is an action-angle form of H_0?
Solution: We simply need to substitute $q(\phi_0, J_0), p(\phi_0, J_0)$ from Eq. (2) into H_0. This gives

$$H_0(\phi_0, J_0) = \omega_0 J_0\,. \tag{6}$$

Since this does not depend on ϕ_0 (it only depends on J_0), it is an action-angle Hamiltonian.

(c) What is the perturbed system H in terms of the old action-angle variables, $H(\phi_0, J_0)$?
Solution: Inserting the transformation into the perturbed Hamiltonian gives

$$H = \omega_0 J_0 + \epsilon H_1\,, \qquad H_1(\phi_0, J_0) = \frac{J_0^2}{m\omega_0^2}\sin^4\phi_0\,, \tag{7}$$

which depends on ϕ_0, so it is no longer in action-angle form.

Now recall that for the perturbed system H, we could find a canonical transformation from (ϕ_0, J_0) to new action-angle variables (ϕ, J). We did this with the type-2 canonical transformation.

(d) How does some type-2 generating function $F_2(\phi_0, J)$ determine the relationship between the old variables (ϕ_0, J_0) and new variables (ϕ, J)? That is, what do the two derivatives $\partial F_2/\partial\phi_0$ and $\partial F_2/\partial J$ yield?

Solution: A type-2 generating function $F_2(\phi_0, J)$ gives the coordinate transformation

$$J_0 = \frac{\partial F_2}{\partial \phi_0}\,, \qquad \phi = \frac{\partial F_2}{\partial J}\,. \tag{8}$$

Specifically, we had the near-identity canonical transformation

$$F_2(\phi_0, J) = \phi_0 J + \epsilon \frac{1}{m\omega_0^2}\frac{J^2}{8\omega_0}\left(2\sin^2\phi_0 + 3\right)\sin\phi_0\cos\phi_0\,. \tag{9}$$

(e) What is the relationship between (ϕ_0, J_0) and (ϕ, J)?

Solution: We need to compute the derivatives in Eq. (9). After a bit of algebra we find

$$J_0 = J + \epsilon\frac{J^2}{2m\omega_0^3}\left(\cos(2\phi_0) - \frac{1}{4}\cos(4\phi_0)\right)\,, \tag{10}$$

$$\phi = \phi_0 + \epsilon\frac{J}{4\omega_0}\left(2\sin^2\phi_0 + 3\right)\sin\phi_0\cos\phi_0\,. \tag{11}$$

Now suppose that ϵ is a time-varying parameter $\epsilon(t)$, which varies on timescales that are very long compared to the oscillation frequency.

(f) What quantity is adiabatically invariant? ***Solution:*** The $O(\epsilon)$-accurate action variable, J, is adiabatically invariant.

(g) Write the adiabatic invariant in terms of the original phase space variables (q, p) using the transformation given in Eq. (2) [Hint 1: In the $\mathcal{O}(\epsilon)$ pieces of the relationship given in 1e, it is consistent to replace J_0 with J or vice versa, which only incurs an error of $\mathcal{O}(\epsilon^2)$. Hint 2: using Eq. (2) to substitute for $\sin\phi_0$ and $\cos\phi_0$ is easier than plugging in some multi-valued function like arctan, as this avoids the need to identify which branch of the function you need]

Solution: First, in the second term on the RHS of Eq. (10), we replace ϵJ^2 with ϵJ_0^2, which incurs a higher-order error of $\mathcal{O}(\epsilon^2)$. This lets us solve for J as

$$J \approx J_0 - \epsilon\frac{J_0^2}{2m\omega_0^3}\left(\cos(2\phi_0) - \frac{1}{4}\cos(4\phi_0)\right)\,. \tag{12}$$

Now we can use multiple-angle formulas to rewrite $\cos(2\phi_0)$ and $\cos(4\phi_0)$ in terms of $\sin\phi_0$ and $\cos\phi_0$, and next eliminate these in favor of q's, p's, and functions of J_0. Finally rewrite J_0 in terms of q's and p's via Eq. (4). After the dust settles, we have

$$J = \frac{p^2}{2m\omega_0} + \frac{1}{2}m\omega_0 q^2 - \epsilon\frac{1}{32m^3\omega_0^5}\left(3p^4 + 6m^2p^2q^2\omega_0^2 - 5m^4q^4\omega_0^4\right)\,. \tag{13}$$

Suppose that at time $t = 0$, $\epsilon(0) = 0$, and there was some maximum oscillation amplitude $q_{\max}$ (at which point the momentum p vanished).

(h) At any time t (or value of ϵ), find an equation that relates $q_{\max}$ (the max displacement, when $p = 0$) and the adiabatic invariant from the previous part.

Solution: We can evaluate Eq. (13) at a turning point, where $p = 0$, calling the value of the turning point $q_{\max}(\epsilon, J)$ or just $q_{\max}$ for short. When we do this we get the relation

$$J = \frac{1}{2}m\omega_0 q_{\max}^2 + \epsilon\frac{5mq_{\max}^4}{32\omega_0}\,. \tag{14}$$

Notice that this is a quadratic in $q_{\max}^2$, so it is actually possible to solve explicitly for $q_{\max}(\epsilon, J)$.

(i) What is the explicit dependence $q_{\max,0}(J)$ when $\epsilon = 0$?

Solution: Plugging in $\epsilon = 0$ we find the simple relationship

$$q_{\max,0}(J) = \sqrt{\frac{2J}{m\omega_0}} \,. \tag{15}$$

Supposing that the *change* in the max displacement is small, you can write the max displacement as $q_{\max} = q_{\max,0} + \epsilon\delta q_{\max}$.

(j) Plugging this approximation into the result from 1h, find an equation for $\delta q_{\max}$, in terms of the original amplitude $q_{\max,0}$.

Solution: Insert this ansatz into Eq. (14). Collect order by order in ϵ, but we discard $\mathcal{O}(\epsilon^2)$ and above, because we have discarded this order in the Hamiltonian and elsewhere. The $\mathcal{O}(\epsilon^0)$ equation is automatically satisfied by construction. At $\mathcal{O}(\epsilon^1)$ we get

$$0 = \frac{1}{2}m\omega_0(2q_{\max,0}\delta q_{\max}) + \frac{5mq^4_{\max,0}}{32\omega_0} \,, \tag{16}$$

which we immediately solve

$$\delta q_{\max} = -\frac{5q^3_{\max,0}}{32\omega_0^2} \,. \tag{17}$$

2. **Cubic correction to the SHO**. Let us now consider a cubic correction to the SHO, by taking the same H_0 as above, but now taking the perturbation

$$H_1 = \frac{1}{3}mq^3 \,. \tag{18}$$

(a) What is the Hamiltonian $H = H_0 + \epsilon H_1$ in terms of the (old) AA vars (ϕ_0, J_0) given previously?

Solution: As before, simply insert Eq. (2), finding

$$H = \omega_0 J_0 + \epsilon H_1 \,, \qquad H_1 = \frac{m}{3}\left(\frac{2J_0}{m\omega_0}\right)^{3/2} \sin^3\phi_0 \,. \tag{19}$$

(b) What are the equations of motion for ϕ_0 and J_0?

Solution: We use Hamilton's equations in terms of ϕ_0, J_0, i.e.

$$\dot\phi_0 = +\frac{\partial H}{\partial J_0} = \omega_0 + \epsilon\frac{m}{3}\left(\frac{2}{m\omega_0}\right)^{3/2}\frac{3}{2}J_0^{1/2}\sin^3\phi_0 \,, \tag{20}$$

$$\dot J_0 = -\frac{\partial H}{\partial \phi_0} = 0 + \epsilon\frac{m}{3}\left(\frac{2J_0}{m\omega_0}\right)^{3/2} 3\sin^2\phi_0\cos\phi_0 \,. \tag{21}$$

Recall that angle-averaging of any quantity f is defined as

$$\langle f(\phi_0, J_0)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\phi_0, J_0)d\phi_0 \,. \tag{22}$$

(c) Average the right-hand-sides of the Hamilton's equations for ϕ_0 and J_0 over a single period of the ϕ_0 motion. In other words, compute $\langle\dot\phi_0\rangle$ and $\langle\dot J_0\rangle$.

Solution: We will need the trigonometric integrals (which can be performed by hand, or using a computer algebra system)

$$\int \sin^3\phi_0\, d\phi_0 = -\frac{3}{4}\cos\phi_0 + \frac{1}{12}\cos 3\phi_0 \,, \tag{23}$$

$$\int \sin^2\phi_0 \cos\phi_0\, d\phi_0 = \frac{1}{3}\sin^3\phi_0 \,. \tag{24}$$

Each of these integrals averages to zero over a cycle, i.e. $\frac{1}{2\pi}\int_0^{2\pi} \cdot \, d\phi_0 = 0$ for both integrands above. Therefore, the secular behavior is unperturbed,

$$\langle \dot{\phi}_0 \rangle = \omega_0 \,, \tag{25}$$

$$\langle \dot{J}_0 \rangle = 0 \,. \tag{26}$$

(d) Also compute the average of the perturbation to the Hamiltonian, $\langle H_1 \rangle$

Solution: We make use of the integral in Eq. (23), which we already found to vanish when averaged over a cycle. Therefore, $\langle H_1 \rangle = 0$.

(e) Comment on the system's secular behavior.

Solution: There is no secular drift.

Now recall that if we want to find a type-2 near-identity generating function to put this system in AA form, we need to compute

$$F_2(\phi_0, J) = \phi_0 J + \epsilon \int^{\phi_0} \frac{\langle H_1 \rangle - H_1(\phi_0', J)}{\omega_0(J)} d\phi_0' \tag{27}$$

(f) Compute the integral above, thus finding the type-2 generating function we need.

Solution: Plugging in we find

$$F_2 = \phi_0 J + \epsilon \frac{1}{\omega_0} \frac{-m}{3} \left(\frac{2J}{m\omega_0} \right)^{3/2} \int^{\phi_0} \sin^3 \phi_0' \, d\phi_0 \,. \tag{28}$$

We already have this integral above in Eq. (23). Therefore the generating function we seek is

$$F_2 = \phi_0 J - \epsilon \frac{m}{3\omega_0} \left(\frac{2J}{m\omega_0} \right)^{3/2} \left(-\frac{3}{4} \cos\phi_0 + \frac{1}{12} \cos 3\phi_0 \right) \tag{29}$$

(g) With this generating function, find the relationship between (ϕ_0, J_0) and (ϕ, J).

Solution: The transformation is found via Eq. (8) (repeated here),

$$J_0 = \frac{\partial F_2}{\partial \phi_0} \,, \qquad \phi = \frac{\partial F_2}{\partial J} \,. \tag{30}$$

In the J_0 equation, taking the ϕ_0 derivative simply undoes the integration that we performed earlier. That is, we get

$$J_0 = J + \epsilon \frac{\langle H_1 \rangle - H_1(\phi_0, J)}{\omega_0(J)} = J - \epsilon \frac{m}{3\omega_0} \left(\frac{2J}{m\omega_0} \right)^{3/2} \sin^3 \phi_0 \,. \tag{31}$$

The ϕ equation is less trivial, but still straightforward to take the derivative,

$$\phi = \phi_0 - \epsilon \frac{m}{3\omega_0} \left(\frac{2}{m\omega_0} \right)^{3/2} \frac{3}{2} J^{1/2} \left(-\frac{3}{4} \cos\phi_0 + \frac{1}{12} \cos 3\phi_0 \right) \,. \tag{32}$$

(h) For good measure: what is a different expression for J in terms of an integral in the original (q, p) phase space variables?

Solution: The $i$the action can be expressed an integral over the ith cycle around the torus (defined by the constants of motion) in phase space,

$$J_i = \frac{1}{2\pi} \int_{\mathcal{C}_i} p_i \, dq_i \qquad \text{(no summation)}. \tag{33}$$

As this is a 1-degree of freedom system, this torus is really just a loop in phase space. The single constant of motion that defines this 1-torus is the energy. So, in fact, this is an integral that follows the particle's trajectory around a curve of constant energy.

If the value of the Hamiltonian is E, then we can solve for $p(q,E)$ as

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2 + \epsilon\frac{m}{3}q^3\,, \tag{34}$$

$$\implies \qquad p^2(q,E) = 2m\left(E - \frac{1}{2}m\omega_0^2 q^2 + \epsilon\frac{m}{3}q^3\right)\,. \tag{35}$$

When the particle is moving to the right, we take the positive root, and when it is moving to the left, we take the negative root.

The integral in Eq. (33) can be rewritten as the sum of two integrals: (1) going from the left turning point $q_-(E)$ to the right turning point $q_+(E)$, and (2) from right to left. In the first integral you take the positive root in Eq. (35), and in the second integral you take the negative root,

$$J(E) = \frac{1}{2\pi}\left[\int_{q_-(E)}^{q_+(E)} p_+(q,E)\,dq + \int_{q_+(E)}^{q_-(E)} p_-(q,E)\,dq\right] \tag{36}$$

But, since the positive and negative roots are simply negatives of each other, these two integrals can be combined into one,

$$J = \frac{\sqrt{2m}}{\pi}\int_{q_-(E)}^{q_+(E)} \sqrt{E - \frac{1}{2}m\omega_0^2 q^2 + \epsilon\frac{m}{3}q^3}\,dq\,. \tag{37}$$

Technically, this integral can be performed analytically in terms of the so-called *complete elliptic integrals*, but it is enough to simply know that the integral exists.