UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 7

Due: Monday, Nov. 5, 2018, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Driven critically damped oscillator**. A critically damped oscillator has $Q = 1/2$. The free oscillator obeys the homogeneous equation of motion $\ddot{q} + 2\dot{q} + q = 0$ (in natural units where $\omega_0 = 1$, or equivalently, using rescaled dimensionless time $\omega_0 t \to t$). The two free oscillator solutions are e^{-t} and te^{-t}.

   (a) Drive this oscillator with an external driving force that is a discontinuous step at $t = 0$: for $t < 0$, $F = 0$, and for $t \geq 0$, $F = 1$. Assuming that $q = \dot{q} = 0$ for $t < 0$, find an explicit solution for $t \geq 0$.

   (b) Consider the oscillatory driving force: for $t < 0$, $F = 0$, and for $t \geq 0$, $F = \cos t$. Again, $q = \dot{q} = 0$ for $t < 0$. Find the form of the steady-state ($t \gg 1$) solution by first solving for $q(t)$ for the complex driving force $F = e^{it}$, $t > 0$, and then finding the physical displacement of the oscillator $q(t)$ for $F(t) = \cos t$. What is the relative phase between the driving force and the oscillator response in the steady state?

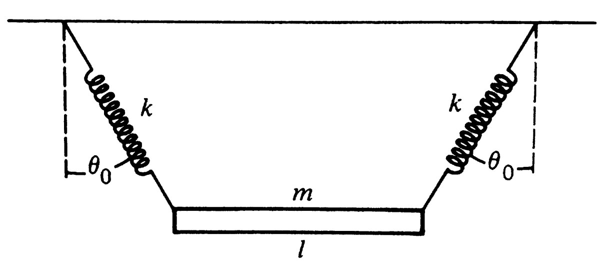
   (c) To find the exact solution for all positive times you could use a Green's function or you could match boundary conditions at $t = 0$. Use the boundary condition method to find the transient solution. Combine this with the result of part (b) to find the oscillator's total response to suddenly turning on a $\cos t$ driving force at $t = 0$. Make a sketch of $q(t)$ for $0 \leq t \leq 4$. For what time is the response maximized?

   (d) The derivative of a step function is a delta function. From this fact, find the response of this oscillator to a delta function impulse at $t = 0$. Then find the explicit form of the Green's function $G(t - t')$. Write the oscillator response to the driving force in part (b), as an integral over t'. What are the limits of integration? It is easy enough to do the Green's function integral, e.g. using Mathematica, so you might want to do this, for no credit, to check the result in part (c).

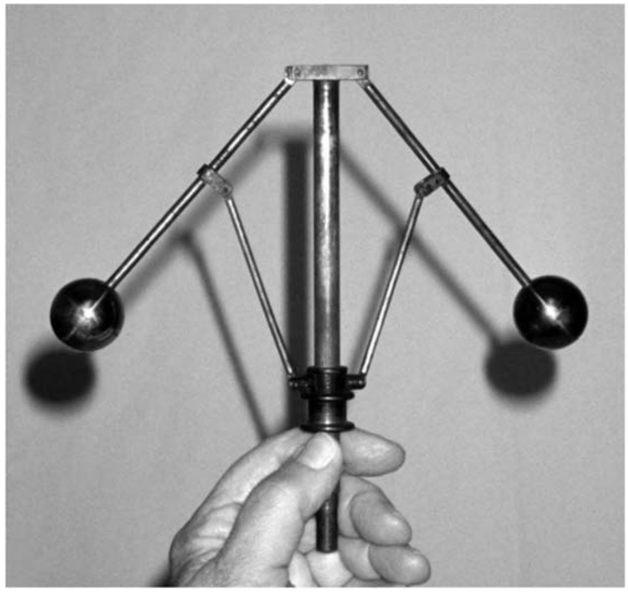
2. Fetter and Walecka problem 4.3. In addition, the following step:

   (g) What is the condition (on the two masses) for the phase-space trajectory to close? Give a fairly simple mass ratio $q = m_1/m_2$ that produces such a closed phase-space orbit.

3. **Plank on springs**. Consider a plank of mass m and length l, which is attached to the ceiling via two springs at its ends. Both springs have the same length d when the system is static, and they have the same spring constant k. Find the normal modes of small oscillation in this 2-dimensional space:



4. **Centrifugal governor**. In a centrifugal governor, the two spheres (each of mass m) are affixed at a distance d from the top of a shaft, on pivots. They are free to pivot vertically, and their angles from the vertical are constrained to be equal via the linkage seen below. They are also constrained to be opposite each other around the shaft. The whole assembly rotates when the shaft is driven at angular velocity Ω.



(a) How many degrees of freedom does this system have? Choose appropriate generalized coordinate(s) and write down a Lagrangian for this system.

(b) As a function of Ω, find the equilibrium (or equilibria) for your generalized coordinate(s). Mention the allowed values of Ω for each equilibrium to be physical.

(c) Linearize about the equilibrium (or equilibria) and find the frequency (or frequencies) of small oscillations. Discuss whether the equilibrium/a are stable or unstable, for different values of Ω.