UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 7 — SOLUTIONS

Due: Monday, Nov. 5, 2018, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Driven critically damped oscillator**. A critically damped oscillator has $Q = 1/2$. The free oscillator obeys the homogeneous equation of motion $\ddot{q} + 2\dot{q} + q = 0$ (in natural units where $\omega_0 = 1$, or equivalently, using rescaled dimensionless time $\omega_0 t \to t$). The two free oscillator solutions are e^{-t} and te^{-t}.

   (a) Drive this oscillator with an external driving force that is a discontinuous step at $t = 0$: for $t < 0$, $F = 0$, and for $t \geq 0$, $F = 1$. Assuming that $q = \dot{q} = 0$ for $t < 0$, find an explicit solution for $t \geq 0$.
   Solution: Taking the particular integral for $t > 0$ to be $q = 1$ the general solution is

   $$q + Ae^{-t} + Bte^{-t} + 1. \tag{1}$$

   The initial conditions at $t = 0$ are $q(0) = 0 \Rightarrow A = -1,\ \dot{q}(0) = 0 \Rightarrow B = -1$, and so the solution is

   $$q(t) = 1 - e^{-t}(t + 1). \tag{2}$$

   Useful checks are $q(t \to \infty) \to 1$ and expanding for small t gives $q \simeq t^2 + \cdots$ which shows the expected acceleration from rest proportional to t^2.

   (b) Consider the oscillatory driving force: for $t < 0$, $F = 0$, and for $t \geq 0$, $F = \cos t$. Again, $q = \dot{q} = 0$ for $t < 0$. Find the form of the steady-state ($t \gg 1$) solution by first solving for $q(t)$ for the complex driving force $F = e^{it}$, $t > 0$, and then finding the physical displacement of the oscillator $q(t)$ for $F(t) = \cos t$. What is the relative phase between the driving force and the oscillator response in the steady state?
   Solution: The equation for the solution with the complex force is

   $$\ddot{q}_c + 2\dot{q}_c + q_c = e^{it}. \tag{3}$$

   The steady state solution is $q_c = q_0 e^{it}$ and then substitution gives

   $$q_0 = -\tfrac{i}{2} = \tfrac{1}{2} e^{-i\pi/2}. \tag{4}$$

   The real part of $q_0 e^{it}$ gives the solution for the cosine forcing

   $$q = \tfrac{1}{2} \sin t \tag{5}$$

   and the phase is $-\pi/2$ relative to the drive.

   (c) To find the exact solution for all positive times you could use a Green's function or you could match boundary conditions at $t = 0$. Use the boundary condition method to find the transient solution. Combine this with the result of part (b) to find the oscillator's total response to suddenly turning on a $\cos t$ driving force at $t = 0$. Make a sketch of $q(t)$ for $0 \leq t \leq 4$. For what time is the response maximized?
   Solution: The full solution to the switched-on cosine force is

   $$q + Ae^{-t} + Bte^{-t} + \tfrac{1}{2} \sin t \tag{6}$$

   and the initial conditions are $q(0) = 0 \Rightarrow A = 0,\ \dot{q}(0) = 0 \Rightarrow B = -\frac{1}{2}$, so that the solution is

   $$q = \tfrac{1}{2}(\sin t - te^{-t}). \tag{7}$$

   The same checks are useful here as well.

(d) The derivative of a step function is a delta function. From this fact, find the response of this oscillator to a delta function impulse at $t = 0$. Then find the explicit form of the Green's function $G(t - t')$. Write the oscillator response to the driving force in part (b), as an integral over t'. What are the limits of integration? It is easy enough to do the Green's function integral, e.g. using Mathematica, so you might want to do this, for no credit, to check the result in part (c).

Solution: Differentiating the result in part (a) gives the result for a step forcing at $t = 0$ and hence, by shifting the pulse to $t = t'$ the Green's function

$$G(t, t') = (t - t')e^{-(t-t')} \quad \text{for } t > t'. \tag{8}$$

The response to the forcing of part (b) is therefore

$$q(t) = \int_0^t (t - t')e^{-(t-t')} \cos t' \, dt' \tag{9}$$

where the lower limit is zero because the force is zero for negative times and $G(t, t')$ is zero for $t < t'$.

To check your result of part (c) you can do the integral using e.g. Mathematica

$$\text{Integrate}\left[(x - y)e^{-(x-y)}\text{Cos}[y], \{y, 0, x\}\right] \quad \Rightarrow \quad \frac{1}{2}\left(-e^{-x}x + \text{Sin}[x]\right) \ . \tag{10}$$

2. Fetter and Walecka problem 4.3.

Solution:

(a) The exact Lagrangian is most easily expressed in terms of the angles θ_1, θ_2 measured from the vertical,

$$L = T - V \,, \tag{11}$$

$$T = \frac{1}{2}m_1 L^2 \dot\theta_1^2 + \frac{1}{2}m_2 L^2 (\dot\theta_1^2 + 2\cos(\theta_1 - \theta_2)\dot\theta_1\dot\theta_2 + \dot\theta_2^2) \,, \tag{12}$$

$$V = -m_1 g L \cos\theta_1 - m_2 g L(\cos\theta_1 + \cos\theta_2) \,. \tag{13}$$

This is easily expanded to second order using $\cos\theta \approx 1 - \theta^2/2$ for $|\theta| \ll 1$. This gives the quadratic Lagrangian $L_q = T_q - V_q$ where

$$T_q = \frac{L^2}{2}\left[(m_1 + m_2)\dot\theta_1^2 + m_2\dot\theta_2^2 + 2m_2\dot\theta_1\dot\theta_2\right] \,, \tag{14}$$

$$V_q = \frac{gL}{2}\left[(m_1 + m_2)\theta_1^2 + m_2\theta_2^2\right] \,, \tag{15}$$

where we have thrown away an irrelevant constant shift of the potential energy.

Now let us convert this to the transverse displacement variables $\eta_1 = L\sin\theta_1$, $\eta_2 = L\sin\theta_2$. Notice that if we expand this in powers of θ_i, we can stop at linear order, because the quadratic and higher pieces will be yet higher order when inserted into the quadratic Lagrangian. Therefore we can just use $\eta_1 \approx L\theta_1$, $\eta_2 \approx L\theta_2$. This is straightforward to put into the Lagrangian and perform a bit of algebra to find the desired result,

$$T_q = \frac{1}{2}\left[m_1\dot\eta_1^2 + m_2(\dot\eta_1 + \dot\eta_2)^2\right] \,, \tag{16}$$

$$V_q = \frac{g}{2L}\left[(m_1 + m_2)\eta_1^2 + m_2\eta_2^2\right] \,. \tag{17}$$

(b) Let us first extract the mass and potential matrices, so that we can write L in the matrix form $L = \frac{1}{2}\dot{\boldsymbol\eta}^T \boldsymbol{M}\dot{\boldsymbol\eta} - \frac{1}{2}\boldsymbol\eta^T\boldsymbol{K}\eta$. The two matrices are

$$\boldsymbol{M} = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \,, \qquad \boldsymbol{K} = \frac{g}{L}\begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix} \tag{18}$$

Now to find the eigenmodes, we want to find where $\det(\omega^2 \boldsymbol{M} - \boldsymbol{K}) = 0$. This gives a quartic in ω that is really a quadratic in ω^2,

$$\det(\omega^2 \boldsymbol{M} - \boldsymbol{K}) = m_1 m_2 \omega^4 - \frac{2gm_2(m_2+m_2)}{L}\omega^2 + \frac{g^2 m_2(m_1+m_2)}{L^2} = 0\,. \tag{19}$$

Now we can solve this with the quadratic equation, and with a bit of manipulation, put this in the desired form

$$\omega^2 = \frac{g}{L}(1 \pm \gamma)^{-1}\,, \qquad \gamma^2 = \frac{m_2}{m_1+m_2}\,. \tag{20}$$

As an aside, note that we can now rewrite $\boldsymbol{M}$ and $\boldsymbol{K}$ as

$$\boldsymbol{M} = (m_1+m_2)\begin{bmatrix} 1 & \gamma^2 \\ \gamma^2 & \gamma^2 \end{bmatrix}, \qquad \boldsymbol{K} = \frac{g(m_1+m_2)}{L}\begin{bmatrix} 1 & 0 \\ 0 & \gamma^2 \end{bmatrix} \tag{21}$$

which makes some of the manipulations simpler.

(c) Now we want to find eigenvectors $\boldsymbol{e}_{(j)}$ for each frequency ω_j that solve

$$\left(\omega_j^2 \boldsymbol{M} - \boldsymbol{K}\right)\boldsymbol{e}_{(j)}\,. \tag{22}$$

Since the eigenvalues are distinct (except in the limit $\gamma \to 0$), each eigenspace should be 1-dimensional, so this is actually simple to solve by hand. Let us assigned $\omega_1^2 = g/L/(1+\gamma)$ and $\omega_2^2 = g/L/(1-\gamma)$, so that $\omega_1 < \omega_2$. Now if we set $e^2_{(j)} = 1$, then we can solve the linear problem by hand to find

$$\boldsymbol{e}_{(1)} = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}, \qquad \boldsymbol{e}_{(2)} = \begin{bmatrix} -\gamma \\ 1 \end{bmatrix}. \tag{23}$$

For the lower frequency mode associated with ω_1, the two particles oscillate in phase with each other, whereas for the higher frequency mode ω_2, they oscillate with opposite phase.

To examine the limits: we can write $\gamma^2 = 1/(1+m_1/m_2)$. If we make $m_1/m_2 \ll 1$, $\gamma^2 \approx 1 - \frac{m_1}{m_2}$ is very close to 1, and $\gamma \approx 1 - \frac{m_1}{2m_2}$. Then the lower oscillation frequency will be close to $\omega_1^2 \approx g/(2L)$. This is like the upper mass, which is negligible, has just become part of a length $2L$ arm for the lower pendulum.

Meanwhile if we take $m_1/m_2 \gg 1$, then $\gamma^2 \approx m_2/m_1 \ll 1$, and the two modes have almost the same frequency, $\omega^2 \approx g(1 \mp \gamma)/L$. Now we can hardly excite motion of the top mass, so the system acts like the bottom mass is a single pendulum with arm length L, and the upper mass barely moves.

(d) The above eigenvectors have not been normalized with $\boldsymbol{M}$ to serve as an inner product. We want normalized eigenvectors $\boldsymbol{a}_{(j)}$ with the condition that $\boldsymbol{a}^T_{(i)}\boldsymbol{M}\boldsymbol{a}_{(j)} = \delta_{ij}$. We do not need the Gram-Schmidt procedure, because our eigenvalues are distinct, so we just need to normalize.

First compute the scalars

$$\boldsymbol{e}^T_{(1)}\boldsymbol{M}\boldsymbol{e}_{(1)} = 2m_2(1+\gamma)\,, \qquad \boldsymbol{e}^T_{(1)}\boldsymbol{M}\boldsymbol{e}_{(2)} = 0\,, \qquad \boldsymbol{e}^T_{(2)}\boldsymbol{M}\boldsymbol{e}_{(2)} = 2m_2(1-\gamma)\,. \tag{24}$$

Thus we should divide by the square roots of these scalars to normalize, giving (after a bit of algebra)

$$\boldsymbol{a}_{(1)} = \frac{\boldsymbol{e}_{(1)}}{\sqrt{2m_2(1+\gamma)}} = \frac{1}{\sqrt{2m_1}}\begin{bmatrix} \sqrt{1-\gamma} \\ \gamma^{-1}\sqrt{1-\gamma} \end{bmatrix} \tag{25}$$

$$\boldsymbol{a}_{(2)} = \frac{\boldsymbol{e}_{(2)}}{\sqrt{2m_2(1-\gamma)}} = \frac{1}{\sqrt{2m_1}}\begin{bmatrix} -\sqrt{1+\gamma} \\ \gamma^{-1}\sqrt{1+\gamma} \end{bmatrix}. \tag{26}$$

These are combined into the modal matrix

$$\boldsymbol{\mathcal{A}} = \frac{1}{\sqrt{2m_1}}\begin{bmatrix} \sqrt{1-\gamma} & -\sqrt{1+\gamma} \\ \gamma^{-1}\sqrt{1-\gamma} & \gamma^{-1}\sqrt{1+\gamma} \end{bmatrix}. \tag{27}$$

Then by construction, we will have

$$\boldsymbol{\mathcal{A}}^T \boldsymbol{M} \boldsymbol{\mathcal{A}} = \boldsymbol{I}\,. \tag{28}$$

We must also check that this diagonalizes $\boldsymbol{K}$. Performing the matrix multiplication shows that this is true,

$$\boldsymbol{\mathcal{A}}^T \boldsymbol{K} \boldsymbol{\mathcal{A}} = \frac{g}{L}\begin{bmatrix}(1+\gamma)^{-1} & 0 \\ 0 & (1-\gamma)^{-1}\end{bmatrix}. \tag{29}$$

(e) The normal coordinates $\boldsymbol{\zeta}$ are related to the displacements $\boldsymbol{\eta}$ via $\boldsymbol{\eta} = \boldsymbol{\mathcal{A}}\boldsymbol{\zeta}$, and the inverse relationship is found with

$$\boldsymbol{\zeta} = \boldsymbol{\mathcal{A}}^T \boldsymbol{M} \boldsymbol{\eta}\,, \tag{30}$$

$$\begin{bmatrix}\zeta_1 \\ \zeta_2\end{bmatrix} = \sqrt{\frac{m_1+m_2}{2}}\begin{bmatrix}\sqrt{1+\gamma} & \gamma\sqrt{1+\gamma} \\ -\sqrt{1-\gamma} & \gamma\sqrt{1-\gamma}\end{bmatrix}\begin{bmatrix}\eta_1 \\ \eta_2\end{bmatrix}. \tag{31}$$

These ζ_i components are decoupled and oscillate independently at frequencies ω_i respectively, so they have general solutions

$$\begin{bmatrix}\zeta_1 \\ \zeta_2\end{bmatrix} = \begin{bmatrix}C_1\cos(\omega_1 t+\phi_1) \\ C_2\cos(\omega_2 t+\phi_2)\end{bmatrix}. \tag{32}$$

(f) For $m_2 \ll m_1$, we have $\gamma^2 \approx m_2/m_1 \ll 1$. Let us call the initial displacement of $\eta_1 = \delta$. Then from Eq. (31), we have (approximately) $\zeta_1 = -\zeta_2 = \delta\sqrt{m_1/2} \equiv \alpha$, so the two normal coordinates are excited approximately equally and with opposite phase. Their solutions will be $\zeta_i = \pm\alpha\cos(\omega_i t)$, where the two frequencies are nearly the same, $\omega_{1,2}^2 \approx g(1\mp\gamma)/L$, so $\omega_{1,2} \approx \sqrt{g/L}(1\mp\gamma/2)$.

Now we recombine the normal coordinates recombine into the displacements via $\boldsymbol{\eta} = \boldsymbol{\mathcal{A}}\boldsymbol{\zeta}$, so we have

$$\begin{bmatrix}\eta_1 \\ \eta_2\end{bmatrix} \approx \delta\begin{bmatrix}\cos(\omega_1 t)+\cos(\omega_2 t) \\ \gamma^{-1}[\cos(\omega_1 t)-\cos(\omega_2 t)]\end{bmatrix} \tag{33}$$

From the standard trig identities this can be expressed as

$$\begin{bmatrix}\eta_1 \\ \eta_2\end{bmatrix} \approx 2\delta\begin{bmatrix}\cos[\frac{1}{2}(\omega_1+\omega_2)t]\cos[\frac{1}{2}(\omega_2-\omega_1)t] \\ \gamma^{-1}\sin[\frac{1}{2}(\omega_1+\omega_2)t]\sin[\frac{1}{2}(\omega_2-\omega_1)t]\end{bmatrix} \tag{34}$$

These solutions are both a rapidly varying carrier with frequency $(\omega_1+\omega_2)/2 \approx \sqrt{g/L}$, and the amplitude of this carrier is being modulated at the frequency difference $(\omega_2-\omega_1) \approx \gamma\sqrt{g/L}$ (this is the frequency with which the amplitude goes from minimum to maximum, not from maximum to maximum).

In addition, the following step:

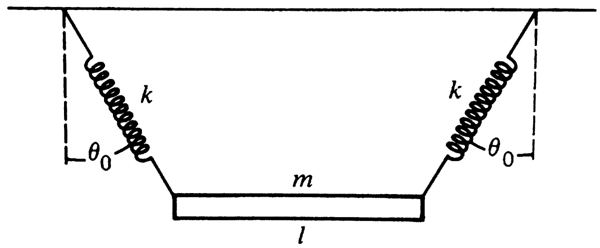
(g) What is the condition (on the two masses) for the phase-space trajectory to close? Give a fairly simple mass ratio $q = m_1/m_2$ that produces such a closed phase-space orbit.

Solution: In order for the phase-space orbit to close, we want the frequency ratio ω_1/ω_2 to be a rational number. Let us write the mass ratio as $q = m_1/m_2$, then the frequency ratio works out to

$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{1-\gamma}{1+\gamma}} = \sqrt{1+2q-2\sqrt{q(1+q)}} \le 1\,. \tag{35}$$

We can find a value of q that makes this a simple ratio, say $\omega_1/\omega_2 = 1/2$. A little algebra gives a unique solution, $q = 9/16$. Of course there are an infinite number of solutions, by the density of the rationals within the reals.

3. **Plank on springs**. Consider a plank of mass m and length l, which is attached to the ceiling via two springs at its ends. Both springs have the same length d when the system is static, and they have the same spring constant k. Find the normal modes of small oscillation in this 2-dimensional space:



Solution: Suppose we choose a 2d coordinate system with the origin $(0,0)$ at the location of the center when in equilibrium. Then the equilibrium left/right ends of the plank are at coordinates $(-\frac{l}{2},0)$ and $(+\frac{l}{2},0)$, respectively. The ceiling is at a height of $d\cos\theta_0$. Thus the left end of the left spring is at $(-\frac{l}{2}-d\sin\theta_0, d\cos\theta_0)$, and the right end of the right spring is at $(\frac{l}{2}+d\sin\theta_0, d\cos\theta_0)$.

There are three degrees of freedom to specify for the configuration of the plank: two translations, and one rotation. Suppose we first translate the center of mass by an amount (x,y), and then we rotate the plank about its center of mass by an angle ϕ. Then the coordinate of its left and right endpoint will be

$$\boldsymbol{r}_L = (x - \tfrac{l}{2}\cos\phi, y - \tfrac{l}{2}\sin\phi)\,, \tag{36}$$

$$\boldsymbol{r}_R = (x + \tfrac{l}{2}\cos\phi, y + \tfrac{l}{2}\sin\phi)\,. \tag{37}$$

This makes it straightforward to compute the squared lengths of the left and right springs,

$$L_L^2 = [x - \tfrac{l}{2}\cos\phi - (-\tfrac{l}{2} - d\sin\theta_0)]^2 + (y - \tfrac{l}{2}\sin\phi - d\cos\theta_0)^2 \tag{38}$$

$$L_R^2 = [x + \tfrac{l}{2}\cos\phi - (+\tfrac{l}{2} + d\sin\theta_0)]^2 + (y + \tfrac{l}{2}\sin\phi - d\cos\theta_0)^2. \tag{39}$$

Now we can assemble the full Lagrangian for this system,

$$L = T - V\,, \qquad T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\phi}^2\,, \qquad V = mgy + \frac{1}{2}k[(L_L - r)^2 + (L_R - r)^2]\,, \tag{40}$$

where r is the *relaxed* length of the springs (different from d), and where I is the moment of inertia of this plank about its center, which is $I = ml^2/12$ for a narrow plank or rod.

So far this is exact. Now we can expand in powers of $\boldsymbol{u} \equiv (x,y,\phi)$. At the equilibrium, the lengths $L_L = d = L_R$, and the potential energy stored in the springs cancels the gravitational potential energy mgy. So, when we expand about the minimum, we only need to keep the quadratic terms in $V_{\min} = \frac{1}{2}k[(L_L - d)^2 + (L_R - d)^2]$.

We can then write the Lagrangian in matrix form as

$$L = \frac{1}{2}\dot{\boldsymbol{u}}^T \boldsymbol{M} \dot{\boldsymbol{u}} - \frac{1}{2}\boldsymbol{u}^T \boldsymbol{K} u\,. \tag{41}$$

Extracting the M and K matrices we find,

$$\boldsymbol{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad \boldsymbol{K} = k \begin{bmatrix} 2\sin^2\theta_0 & 0 & l\cos\theta_0\sin\theta_0 \\ 0 & 2\cos^2\theta_0 & 0 \\ l\cos\theta_0\sin\theta_0 & 0 & \frac{1}{2}l^2\cos^2\theta_0 \end{bmatrix} \tag{42}$$

We can then find the eigenvalues and eigenvectors of $\boldsymbol{\Lambda} \equiv \boldsymbol{M}^{-1}\boldsymbol{K}$ to find the normal mode frequencies and mode shapes. This can be done with e.g. Mathematica. The three normal mode frequencies are

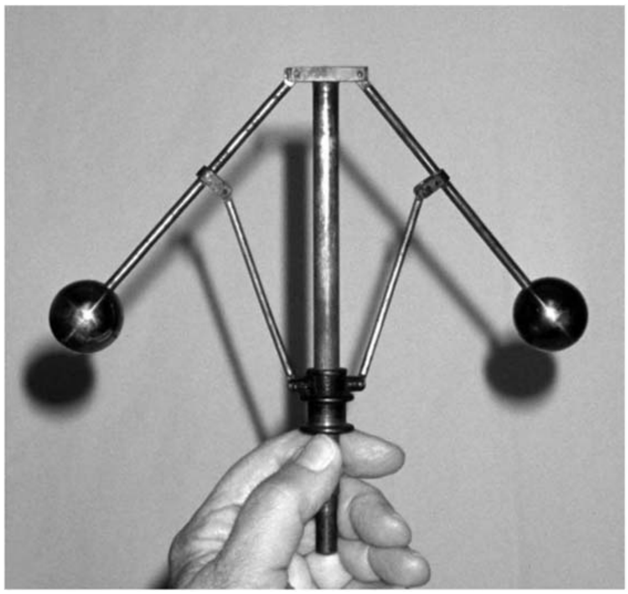
$$\omega_1^2 = 0\,, \qquad \omega_2^2 = \frac{2k}{m}\cos^2\theta_0\,, \qquad \omega_3^2 = \frac{2k}{m}(1 + 2\cos^2\theta_0)\,. \tag{43}$$

Meanwhile the three corresponding mode shapes are given by

$$\boldsymbol{e}_1 = \begin{bmatrix} -l\cos\theta_0 \\ 0 \\ 2\sin\theta_0 \end{bmatrix}, \qquad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{e}_3 = \begin{bmatrix} l\sin\theta_0 \\ 0 \\ 6\cos\theta_0 \end{bmatrix}. \tag{44}$$

We see a zero mode here which involves a horizontal displacement and a rotation of the plank. Of course if we try to make a very large amplitude oscillation in this mode, there will eventually be a restoring force—only it is nonlinear (it is actually quartic in the potential, meaning there is a cubic restoring force for this mode).

4. **Centrifugal governor**. In a centrifugal governor, the two spheres (each of mass m) are affixed at a distance d from the top of a shaft, on pivots. They are free to pivot vertically, and their angles from the vertical are constrained to be equal via the linkage seen below. They are also constrained to be opposite each other around the shaft. The whole assembly rotates when the shaft is driven at angular velocity Ω.



(a) How many degrees of freedom does this system have? Choose appropriate generalized coordinate(s) and write down a Lagrangian for this system.

Solution: Suppose we use a spherical polar coordinate system with the origin at the top of the governor. Each sphere is described by coordinates $(r, \theta, \phi)_{1,2}$ (but let us measure θ from the negative z axis). The r_i coordinates for both spheres is constrained to be a constant d; and since the governor is driven at a constant angular frequency, the ϕ_i coordinate of each sphere is also constrained, $\phi_i = \Omega t + c_i$ (where $c_2 = c_1 + \pi$). Therefore each ball only has one degree of freedom, θ_i. But since there is a linkage between the two balls, $\theta \equiv \theta_1 = \theta_2$, and overall there is only one degree of freedom.

The Lagrangian is given by $L = T - V$ where

$$T = md^2\left(\dot{\theta}^2 + \sin^2\theta\ \Omega^2\right) \tag{45}$$

$$V = -2mgd\cos\theta\,. \tag{46}$$

(b) As a function of Ω, find the equilibrium (or equilibria) for your generalized coordinate(s). Mention the allowed values of Ω for each equilibrium to be physical.

Solution: This can be treated as a one-dimensional particle moving in an effective potential. Divide the Lagrangian by $2md^2$ (this does nothing to the equations of motion). Then we have the new Lagrangian

$$L' = \frac{1}{2}\dot{\theta}^2 - V_{\text{eff}}(\theta)\,, \qquad V_{\text{eff}}(\theta) = -\frac{1}{2}\Omega^2\sin^2\theta - \frac{g}{d}\cos\theta\,. \tag{47}$$

The first term is concave down for all values of Ω, whereas the second term is concave up.

To find the equilibrium points, we must solve for values of θ such that

$$0 = \frac{dV_{\text{eff}}}{d\theta} = -\Omega^2 \sin\theta\cos\theta + \omega_0^2 \sin\theta \tag{48}$$

where we have defined $\omega_0^2 \equiv g/d$.

Two solutions are where $\sin\theta = 0$, that is, $\theta_1 = 0$ and $\theta_2 = \pi$ (our intuition says that the first should be stable and the second unstable, but we will find out for sure in the next part). If $\sin\theta \neq 0$ then we can divide through and find a third solution where

$$\cos\theta_3 = \frac{\omega_0^2}{\Omega^2}\,. \tag{49}$$

Clearly this latter solution only exists if $\omega_0^2/\Omega^2 \leq 1$, because $-1 \leq \cos\theta \leq +1$.

(c) Linearize about the equilibrium (or equilibria) and find the frequency (or frequencies) of small oscillations. Discuss whether the equilibrium/a are stable or unstable, for different values of Ω.

Solution: Since this is a one-dimensional problem, at some extremum θ_i, the frequency of small oscillations is given by $\omega_i^2 = d^2V_{\text{eff}}/d\theta^2|_{\theta_i}$. The second derivative is

$$\frac{d^2V_{\text{eff}}}{d\theta^2} = -\Omega^2\cos 2\theta + \omega_0^2\cos\theta\,. \tag{50}$$

Evaluating this at the three possible equilibria $\theta_1, \theta_2, \theta_3$, we find

$$\omega_1^2 = +\omega_0^2 - \Omega^2\,, \tag{51}$$

$$\omega_2^2 = -\omega_0^2 - \Omega^2\,, \tag{52}$$

$$\omega_3^2 = \Omega^2\left(1 - \frac{\omega_0^4}{\Omega^4}\right)\,. \tag{53}$$

Thus the equilibrium $\theta_1 = 0$ is stable so long as $\Omega^2 < \omega_0^2$. The equilibrium $\theta_2 = \pi$ is always unstable. And the equilibrium $\theta_3 = \arccos(\omega_0^2/\Omega^2)$ only exists when $\Omega^2 \geq \omega_0^2$, in which case it is stable.