**UNIVERSITY OF MISSISSIPPI**
Department of Physics and Astronomy
Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

**Problem Set 5**

**Due**: Friday, Oct. 12, 2018, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Effective 1D analysis of the heavy symmetric top**. After a certain amount of algebra, the *nutation* motion (in the $\theta$ direction) of a heavy symmetric top on a pivot can be reduced to the conservation of a (shifted) energy,

$$E' = \frac{I_1}{2}\dot{\theta}^2 + \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mg\ell\cos\theta\,, \tag{1}$$

where $E' = E - I_3\omega_3^2$ is the shifted energy, $a = I_3\omega_3/I_1$, $b = p_\phi/I_1$, and $\omega_3$ is the angular frequency about the 3 axis.

(a) Change variables to $u = \cos\theta$ and solve for $\dot{u}^2 = f(u)$. What is $f(u)$? (You should get a cubic polynomial in $u$).

(b) From the sign of the leading coefficient of $u^3$ in $f(u)$, does $f(u)$ go to positive or negative infinity as $u \to +\infty$?

(c) What is the value of $f(0)$? What about $f(\pm 1)$? Does $f(\pm 1)$ have a definite sign (always positive, always negative, or does it depend)?

(d) Physical motion is only possible if $\dot{u}$ is real, therefore where $f(u)$ is positive. Argue from this and the results of the previous steps that $f(u)$ has three real roots.

2. **Angular momentum in 4 dimensions**. Suppose that we have a rotationally-invariant 4-dimensional Lagrangian,

$$L = \frac{1}{2}m\dot{\boldsymbol{r}}^2 - V(r)\,, \tag{2}$$

where $\boldsymbol{r} = (x, y, z, w)$ is a 4-vector.

(a) We want to perform an infinitesimal 4-dimensional rotation on $\boldsymbol{r}$. How many parameters are there to specify a 4-d rotation? [Hint: it is only in three dimensions that every rotation can be considered as being "about a fixed axis". More generally, rotations take place in (sets of) two-planes] Give a complete basis for these infinitesimal rotations.

(b) Pick one of these infinitesimal rotations, e.g. the one which generates a rotation in the $z - w$ plane. What are the infinitesimal transformations $\delta r^i$ for this infinitesimal rotation? What about the infinitesimal change to the velocities, $\delta\dot{r}^i$? What is the infinitesimal transformation of the Lagrangian, $\delta L$?

(c) What is the *Noether current* (conserved quantity) which is generated by the above transformation?

3. **From Lagrangian to Hamiltonian**. Consider the 3-dimensional Lagrangian system

$$L = -m\sqrt{1 - \dot{\boldsymbol{r}}^2} - V(r)\,. \tag{3}$$

What are the canonical momenta? Perform the Legendre transform to construct the Hamiltonian for this system. You must be able to write the Hamiltonian in terms of the momenta only, no velocities.

4. **Practice with Poisson brackets**. Consider the Hamiltonian

$$H(q^1, q^2, p_1, p_2) = q^1 p_1 - q^2 p_2 - (q^1)^2 \, a + (q^2)^2 \, b \tag{4}$$

where $a, b$ are some real constants. Show that the following functions are all constants of motion:

$$f_1 = \frac{p_2 - bq^2}{q^1} \tag{5}$$

$$f_2 = q^1 q^2 \tag{6}$$

$$f_3 = q^1 e^{-t} \, . \tag{7}$$

[Hint: recall that for any function $f$, $\frac{d}{dt} f = \frac{\partial}{\partial t} f + \{f, H\}$, where $\{\cdot, \cdot\}$ is the Poisson bracket.]

5. **(Extra credit) Proving the Jacobi identity**. Consider a $2n$-dimensional phase space, and let $f, g, h$ each be functions on phase space. Prove the Jacobi identity,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{8}$$

[Hint: this is easiest if you combine the $q^i, p_j$ coordinates on phase space into the unified coordinate $\eta^i$, use the symplectic matrix $\boldsymbol{J}$, and remember that mixed partial derivatives commute. This last point is especially important to realize that quantities such as $\frac{\partial^2}{\partial \eta^i \partial \eta^j} f$, when treated as a matrix, is symmetric.]