UNIVERSITY OF MISSISSIPPI
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Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 5 — SOLUTIONS

Due: Friday, Oct. 12, 2018, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Effective 1D analysis of the heavy symmetric top**. After a certain amount of algebra, the *nutation* motion (in the θ direction) of a heavy symmetric top on a pivot can be reduced to the conservation of a (shifted) energy,

$$E' = \frac{I_1}{2}\dot{\theta}^2 + \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mg\ell\cos\theta\,, \tag{1}$$

where $E' = E - I_3\omega_3^2$ is the shifted energy, $a = I_3\omega_3/I_1$, $b = p_\phi/I_1$, and ω_3 is the angular frequency about the 3 axis.

(a) Change variables to $u = \cos\theta$ and solve for $\dot{u}^2 = f(u)$. What is $f(u)$? (You should get a cubic polynomial in u).
Solution: We use $\sin^2\theta = 1 - \cos^2\theta = 1 - u^2$, and the chain rule for the derivative,

$$\frac{du}{dt} = -\sin\theta\dot{\theta} = \mp\sqrt{1-u^2}\dot{\theta}\,. \tag{2}$$

Now replace all functions of θ with functions of u in Eq. (1).
Now commence a bit of algebraic simplification. Introduce the two positive constants $\alpha \equiv 2E'/I_1$ and $\beta = 2Mg\ell/I_1$, we can write $f(u)$ as

$$\boxed{f(u) = (1-u^2)(\alpha - \beta u) - (b - au)^2}\,. \tag{3}$$

(b) From the sign of the leading coefficient of u^3 in $f(u)$, does $f(u)$ go to positive or negative infinity as $u \to +\infty$?
Solution: The leading coefficient is β, and $\beta > 0$, so $f(u) \to +\infty$ as $u \to +\infty$.

(c) What is the value of $f(0)$? What about $f(\pm 1)$? Does $f(\pm 1)$ have a definite sign (always positive, always negative, or does it depend)?
Solution: At $u = 0$, we have $\boxed{f(0) = \alpha - b^2}$. At $u = \pm 1$, we have $\boxed{f(\pm 1) = -(b \mp a)^2}$. Since it is the negative of a square, it is never positive (it can be negative, or it can be zero if $a = \pm b$).

(d) Physical motion is only possible if $\dot{u}$ is real, therefore where $f(u)$ is positive. Argue from this and the results of the previous steps that $f(u)$ has three real roots.
Solution: As we found before, $f(+1)$ is negative (we are temporarily ignoring the set of measure 0 where $a = \pm b$), but $f(u) \to +\infty$ as $u \to \infty$. Therefore there is at least one real root at a value of u greater than 1. However, we must have $f(u)$ positive between $-1 \le u \le +1$ in order for physical motion to take place. Since it is negative at both endpoints and positive somewhere in between, there are two additional roots, both inside the interval $-1 \le u \le +1$. (We have ignored the possibility that the cubic has a double root, rather than two simple roots, in this range).

2. **Angular momentum in 4 dimensions**. Suppose that we have a rotationally-invariant 4-dimensional Lagrangian,

$$L = \frac{1}{2} m \dot{\boldsymbol{r}}^2 - V(r)\,, \tag{4}$$

where $\boldsymbol{r} = (x, y, z, w)$ is a 4-vector.

(a) We want to perform an infinitesimal 4-dimensional rotation on $\boldsymbol{r}$. How many parameters are there to specify a 4-d rotation? [Hint: it is only in three dimensions that every rotation can be considered as being "about a fixed axis". More generally, rotations take place in (sets of) two-planes] Give a complete basis for these infinitesimal rotations.

Solution: A rotation in D dimensions can take place in any 2-plane. Thus the number of parameters is given by choosing 2 out of D directions, i.e. $\binom{D}{2} = \frac{D(D-1)}{2}$. For $D = 4$, this is 6 possible rotations.

An infinitesimal rotation is given by $\mathbf{1} + \epsilon \boldsymbol{m}$, and we can show that the $\boldsymbol{m}$ matrix must be antisymmetric for this to generate an orthogonal transformation. Thus we seek a basis for 4×4 antisymmetric matrices. Such a basis is given by e.g. the six matrices

$$\boldsymbol{e}_{xy} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{e}_{xz} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{e}_{xw} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

$$\boldsymbol{e}_{yz} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{e}_{yw} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{e}_{zw} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{6}$$

The subscripts on the $\boldsymbol{e}$ matrices are not indices, they are merely labeling the basis elements.

(b) Pick one of these infinitesimal rotations, e.g. the one which generates a rotation in the $z - w$ plane. What are the infinitesimal transformations δr^i for this infinitesimal rotation? What about the infinitesimal change to the velocities, $\delta \dot{r}^i$? What is the infinitesimal transformation of the Lagrangian, δL?

Solution: Suppose we make the transformation

$$\boldsymbol{r}' = (\mathbf{1} + \epsilon \boldsymbol{e}_{zw}) \boldsymbol{r} = \boldsymbol{r} + \epsilon \boldsymbol{\delta r}\,. \tag{7}$$

Then we have that $\boldsymbol{\delta r} = \boldsymbol{e}_{zw} \boldsymbol{r}$, so we find

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w \\ -z \end{bmatrix}. \tag{8}$$

Taking the time derivative, $\delta \dot{x} = 0, \delta \dot{y} = 0, \delta \dot{z} = \dot{w}$ and $\delta \dot{w} = -\dot{z}$.

If we write $L' = L + \epsilon \delta L$, then we find the infinitesimal transformation to the Lagrangian is

$$\delta L = m \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{\delta r}} = m(\dot{z}\dot{w} - \dot{w}\dot{z}) = 0\,. \tag{9}$$

(c) What is the *Noether current* (conserved quantity) which is generated by the above transformation?

Solution: When a Lagrangian transforms such that $\delta L = 0$ or even $\delta L = \frac{d}{dt} \delta \Phi$, Noether's theorem tells us that the conserved current under such a transformation is given by

$$J = \frac{\partial L}{\partial \dot{q}^i} \delta q^i + \delta \Phi\,. \tag{10}$$

Above, we had $\delta\Phi = 0$. The momenta are given by $\partial L/\partial \dot{r}^i = m\dot{r}^i$. Contracting this with the δr^i we found before, we get the conserved current

$$\boxed{J_{zw} = m(z\dot{w} - w\dot{z})}\,. \tag{11}$$

There are corresponding conserved currents for the other 5 generators.

3. **From Lagrangian to Hamiltonian**. Consider the 3-dimensional Lagrangian system

$$L = -m\sqrt{1-\dot{\boldsymbol{r}}^2} - V(r)\,. \tag{12}$$

What are the canonical momenta? Perform the Legendre transform to construct the Hamiltonian for this system. You must be able to write the Hamiltonian in terms of the momenta only, no velocities.

Solution: Computing the canonical momenta,

$$p_i = \frac{\partial L}{\partial \dot{r}^i} = \frac{m\dot{r}_i}{\sqrt{1-\dot{\boldsymbol{r}}^2}}\,. \tag{13}$$

Now we would like to solve for $\dot{r}^i$ in terms of p_i, so that we will be able to eliminate the velocities from the Hamiltonian. To do so, consider $\boldsymbol{p}^2 = p_i p^i$,

$$\boldsymbol{p}^2 = \frac{m^2\dot{\boldsymbol{r}}^2}{1-\dot{\boldsymbol{r}}^2}\,. \tag{14}$$

Now we can solve for $\dot{\boldsymbol{r}}^2$ in terms of $\boldsymbol{p}^2$,

$$\dot{\boldsymbol{r}}^2 = \frac{\boldsymbol{p}^2}{m^2+\boldsymbol{p}^2}\,. \tag{15}$$

Now assemble the Hamiltonian,

$$H = p_i\dot{r}^i - L = \frac{m\dot{\boldsymbol{r}}^2}{\sqrt{1-\dot{\boldsymbol{r}}^2}} + m\sqrt{1-\dot{\boldsymbol{r}}^2} + V(r)\,. \tag{16}$$

Now use Eq. (15) to eliminate $\dot{\boldsymbol{r}}^2$. Simplifying, we find

$$\boxed{H = \sqrt{m^2+\boldsymbol{p}^2} + V(r)}\,. \tag{17}$$

4. **Practice with Poisson brackets**. Consider the Hamiltonian

$$H(q^1,q^2,p_1,p_2) = q^1p_1 - q^2p_2 - (q^1)^2\,a + (q^2)^2\,b \tag{18}$$

where a, b are some real constants. Show that the following functions are all constants of motion:

$$f_1 = \frac{p_2 - bq^2}{q^1} \tag{19}$$

$$f_2 = q^1q^2 \tag{20}$$

$$f_3 = q^1e^{-t}\,. \tag{21}$$

[Hint: recall that for any function f, $\frac{d}{dt}f = \frac{\partial}{\partial t}f + \{f,H\}$, where $\{\cdot,\cdot\}$ is the Poisson bracket.]

Solution: This is a straightforward application of the definition of the Poisson bracket,

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\,. \tag{22}$$

Alternatively, we can treat this as an application of the rules that Poisson brackets are $\mathbb{R}-$linear in each slot and are derivations in each slot, so they satisfy a chain rule and a Leibniz rule; this allows one to expand the brackets out until we are left with combinations such as $\{q^1, q^2\}$ or $\{p_2, q^2\}$. These basic Poisson brackets are given by $\{q^i, p_j\} = \delta^i_j = -\{p_j, q^i\}$ and all others (q's with q's, p's with p's) vanishing.

Let us first evaluate

$$\{q^1, H\} = \{q^1, q^1 p_1\} = q^1 \qquad \text{(only the term containing } p_1 \text{ can contribute)} \tag{23}$$
$$\{p_1, H\} = \{p_1, q^1 p_1 - (q^1)^2 a\} = -p_1 + 2q^1 a \qquad \text{(only the terms containing } q^1 \text{ can contribute)} \tag{24}$$
$$\{q^2, H\} = \{q^2, -q^2 p_2\} = -q^2 \qquad \text{(only the term containing } p_2 \text{ can contribute)} \tag{25}$$
$$\{p_2, H\} = \{p_2, -q^2 p_2 + (q^2)^2 b\} = p_2 - 2q^2 b \qquad \text{(only the terms containing } q^2 \text{ can contribute)} \tag{26}$$

Now we expand the total time derivatives of the f's, and plug in these results. Only f_3 has explicit time dependence.

(a) f_1:

$$\frac{d}{dt} f_1 = \left\{ \frac{p_2 - bq^2}{q^1}, H \right\} \tag{27}$$
$$= \frac{1}{q^1}\{p_2, H\} - \frac{b}{q^1}\{q^2, H\} - \frac{p_2 - bq^2}{(q^1)^2}\{q^1, H\} = 0\,. \tag{28}$$

(b) f_2:

$$\frac{d}{dt} f_2 = \{q^1 q^2, H\} = \{q^1, H\} q^2 + q^1 \{q^2, H\} = 0 \tag{29}$$

(c) f_3:

$$\frac{d}{dt} f_3 = \frac{\partial}{\partial t} q^1 e^{-t} + \{q^1 e^{-t}, H\} = -q^1 e^{-t} + e^{-t}\{q^1, H\} = 0 \tag{30}$$

5. **(Extra credit) Proving the Jacobi identity**. Consider a $2n$-dimensional phase space, and let f, g, h each be functions on phase space. Prove the Jacobi identity,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{31}$$

[Hint: this is easiest if you combine the q^i, p_j coordinates on phase space into the unified coordinate η^i, use the symplectic matrix $\boldsymbol{J}$, and remember that mixed partial derivatives commute. This last point is especially important to realize that quantities such as $\frac{\partial^2}{\partial\eta^i \partial\eta^j} f$, when treated as a matrix, is symmetric.]

Solution: Recall that in terms of the unified coordinate η^i, the Poisson bracket is

$$\{f, g\} = \frac{\partial f}{\partial \eta^i} J^{ij} \frac{\partial g}{\partial \eta^j} \tag{32}$$

where $J^{ij} = \begin{bmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix}$. The first term in Eq. (31) expands out to

$$\{f, \{g, h\}\} = \left\{ f, \frac{\partial g}{\partial \eta^k} J^{kl} \frac{\partial h}{\partial \eta^l} \right\} = \left\{ f, \frac{\partial g}{\partial \eta^k} \right\} J^{kl} \frac{\partial h}{\partial \eta^l} + \frac{\partial g}{\partial \eta^k} J^{kl} \left\{ f, \frac{\partial h}{\partial \eta^l} \right\} \tag{33}$$

(the term with $\{f, J^{kl}\}$ vanishes because J is a constant matrix). Continuing,

$$\{f, \{g, h\}\} = \frac{\partial f}{\partial \eta^i} J^{ij} \frac{\partial^2 g}{\partial \eta^j \partial \eta^k} J^{kl} \frac{\partial h}{\partial \eta^l} + \frac{\partial g}{\partial \eta^k} J^{kl} \frac{\partial f}{\partial \eta^i} J^{ij} \frac{\partial^2 h}{\partial \eta^j \partial \eta^l}\,. \tag{34}$$

Here we see that there is one term with a second derivative of g, one term with a second derivative of h, and no terms with second derivatives of f. All indices are dummy indices, so we are free to relabel them, e.g. so that the indices of the second derivative (Hessian) matrix are always i and j. We can also use $J^{ij} = -J^{ij}$ (antisymmetry of J) to make this look more like matrix/vector products:

$$\{f, \{g, h\}\} = \frac{\partial f}{\partial \eta^a} J^{ai} \frac{\partial^2 g}{\partial \eta^i \partial \eta^j} J^{jb} \frac{\partial h}{\partial \eta^b} - \frac{\partial g}{\partial \eta^a} J^{ai} \frac{\partial^2 h}{\partial \eta^i \partial \eta^j} J^{jb} \frac{\partial f}{\partial \eta^b} \,. \tag{35}$$

To get the remaining two terms from Eq. (31), we can permute $f \to g \to h \to f$ one and two times. We will get two terms with second derivatives of each of f, g, and h.

$$\begin{aligned}\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = &\frac{\partial f}{\partial \eta^a} J^{ai} \frac{\partial^2 g}{\partial \eta^i \partial \eta^j} J^{jb} \frac{\partial h}{\partial \eta^b} - \frac{\partial g}{\partial \eta^a} J^{ai} \frac{\partial^2 h}{\partial \eta^i \partial \eta^j} J^{jb} \frac{\partial f}{\partial \eta^b} \\ &+ \frac{\partial g}{\partial \eta^a} J^{ai} \frac{\partial^2 h}{\partial \eta^i \partial \eta^j} J^{jb} \frac{\partial f}{\partial \eta^b} - \frac{\partial h}{\partial \eta^a} J^{ai} \frac{\partial^2 f}{\partial \eta^i \partial \eta^j} J^{jb} \frac{\partial g}{\partial \eta^b} \\ &+ \frac{\partial h}{\partial \eta^a} J^{ai} \frac{\partial^2 f}{\partial \eta^i \partial \eta^j} J^{jb} \frac{\partial g}{\partial \eta^b} - \frac{\partial f}{\partial \eta^a} J^{ai} \frac{\partial^2 g}{\partial \eta^i \partial \eta^j} J^{jb} \frac{\partial h}{\partial \eta^b} \,. \end{aligned} \tag{36}$$

Now inspect the terms, pairing them up by which Hessian matrix appears. You will find that every term appears with its negative. Thus we have proven Eq. (31).