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Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 4 — SOLUTIONS

Due: Wednesday, Sep. 26, 2018, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Commutators from Lie groups to Lie algebras**. Let G be some Lie group, which is not necessarily commutative, for example a group of matrices. The associated Lie algebra, $\mathfrak{g}$, is the space of infinitesimal transformations in the neighborhood of the identity element $\mathbb{1}$. That is, if we take the group element $A \in G$ given by $A = \mathbb{1} + \epsilon a$,

$$a = \frac{d}{d\epsilon} A \bigg|_{\epsilon=0} \tag{1}$$

then $a \in \mathfrak{g}$ is in the Lie algebra, and we say that a "generates" A.

(a) Find the inverse element A^{-1} up to order ϵ^2.

Solution: Pose as an ansatz for the inverse

$$A^{-1} = \mathbb{1} + \epsilon a_1 + \epsilon^2 a_2 \,. \tag{2}$$

From the product AA^{-1} and collect the terms order by order in ϵ,

$$AA^{-1} = (\mathbb{1} + \epsilon a)(\mathbb{1} + \epsilon a_1 + \epsilon^2 a_2) \tag{3}$$

$$\mathbb{1} = \mathbb{1} + \epsilon(a + a_1) + \epsilon^2(a a_1 + a_2) + \mathcal{O}(\epsilon^3) \,. \tag{4}$$

Each order in ϵ must satisfy the equality. Therefore we have the two equations $a + a_1 = 0$ and $a a_1 + a_2$. The first equation is solved via $a_1 = -a$. Plugging this into the second we find $a_2 = +a^2$. Therefore we found

$$\boxed{A^{-1} = \mathbb{1} - \epsilon a + \epsilon^2 a^2 + \mathcal{O}(\epsilon^3)} \,. \tag{5}$$

(b) The commutator of two group elements $A, B \in G$ is given by the (non-commutative) product $ABA^{-1}B^{-1}$. Take A, B to be generated by a, b respectively. Expand out the commutator $ABA^{-1}B^{-1}$ up to order ϵ^2.

Solution: We are posing $A = \mathbb{1} + \epsilon a$ and $B = \mathbb{1} + \epsilon b$, so their inverses are $A^{-1} = \mathbb{1} - \epsilon a + \epsilon^2 a^2 + \mathcal{O}(\epsilon^3)$ and $B^{-1} = \mathbb{1} - \epsilon b + \epsilon^2 b^2 + \mathcal{O}(\epsilon^3)$. Now we want to evaluate

$$ABA^{-1}B^{-1} = (\mathbb{1} + \epsilon a)(\mathbb{1} + \epsilon b)(\mathbb{1} - \epsilon a + \epsilon^2 a^2)(\mathbb{1} - \epsilon b + \epsilon^2 b^2) + \mathcal{O}(\epsilon^3) \,. \tag{6}$$

It is straightforward to multiply out all the terms. The only thing to be careful of is that the algebra elements need not commute, so $ab \neq ba$. The result is

$$ABA^{-1}B^{-1} = \mathbb{1} + \epsilon^2(ab - ba) + \epsilon^3 \tag{7}$$

$$\boxed{ABA^{-1}B^{-1} = \mathbb{1} + \epsilon^2[a, b] + \epsilon^3} \tag{8}$$

where we have defined the Lie algebra commutator $[a, b] = ab - ba$.

2. **A few moments of inertia**.

(a) Consider a rectangular prism of uniform density ρ with side lengths a, b, c, centered at the origin, aligned with the (x, y, z) axes. Compute the moment of inertia tensor I_{ij}.

Solution: Recall that the moment of inertia tensor is defined as

$$I_{ij} \equiv \int \rho(r^2\delta_{ij} - r_i r_j)d^3x \,. \tag{9}$$

Let us define the *second moment of mass-density*,

$$m_{ij} \equiv \int \rho r_i r_j d^3x \,. \tag{10}$$

We can see that the trace is ${m^k}_k = \int \rho r^2 d^3x$. Therefore we can compute I_{ij} as

$$I_{ij} = \delta_{ij}{m^k}_k - m_{ij} \,. \tag{11}$$

This type of operation is sometimes called a "trace adjustment."

Now, let us evaluate the tensor m_{ij}. Writing it out in terms of 1-dimensional integrals we have (since density is constant)

$$m_{ij} = \rho \int_{-a/2}^{+a/2} \int_{-b/2}^{+b/2} \int_{-c/2}^{+c/2} r_i r_j \, dz \, dy \, dx \,. \tag{12}$$

Notice that if $i \neq j$, the integrals over r^i and r^j are integrating a linear function over a symmetric integral. For example, if $i = 1$ and $j = 2$, then the x integral is $\int_{-a/2}^{+a/2} x dx = 0$. Therefore we can only get nonvanishing tensor coefficients if $i = j$.

Let us evaluate the m_{xx} coefficient. The y and z integrals can be done immediately, leaving

$$m_{xx} = \rho bc \int_{-a/2}^{+a/2} x^2 dx = \rho bc \left[\frac{x^3}{3}\right]_{x=-a/2}^{x=+a/2} = \frac{1}{12}\rho a^3 bc \,. \tag{13}$$

A quick sanity check: the units of m_{ij} ought to be mass×length^2, so this works out.

The other two diagonal components proceed the same way, so we have

$$m_{ij} = \frac{1}{12}\rho abc \, \mathrm{diag}(a^2, b^2, c^2) \,. \tag{14}$$

The trace is simply ${m^k}_k = \frac{1}{12}\rho abc(a^2 + b^2 + c^2)$.

Finally we can combine these results via Eq. (11). We get

$$\boxed{I_{ij} = \frac{1}{12}\rho abc \, \mathrm{diag}(b^2 + c^2, a^2 + c^2, a^2 + b^2)} \,. \tag{15}$$

(b) Now suppose that we rotate the shape in the x–y plane by 45°. What are two different ways to compute the moment of inertia tensor of the rotated prism? What is the new tensor $I_{i'j'}$?

Solution: The first way to compute the moment of inertia tensor for the rotated solid is to simply perform the straightforward integrations as before, for the rotated solid which lies between $x + y \geq -a/\sqrt{2}$, $x + y \leq +a/\sqrt{2}$, $x - y \leq -b/\sqrt{2}$, $x - y \geq +b/\sqrt{2}$. However, this is quite cumbersome.

The second approach is to take the result from the previous part, and use how tensors transform under a linear transformation. Recall that if you make a transformation such that $v^{i'} = {M^{i'}}_k v^k$ with some transformation ${M^{i'}}_k$, then a tensor will transform as

$$I^{i'j'} = {M^{i'}}_k I^{kl} {M^{j'}}_l \,. \tag{16}$$

In matrix-vector language, if we have $\boldsymbol{v}' = \boldsymbol{M}\boldsymbol{v}$, then the tensor (treated as a matrix) transforms as $\boldsymbol{I}' = \boldsymbol{M}^T \boldsymbol{I} \boldsymbol{M}$.

For our purposes, we want a rotation of 45° about the z axis, which is given by the matrix

$$\boldsymbol{M} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{17}$$

Applying the rotation via $\boldsymbol{I}' = \boldsymbol{M}^T \boldsymbol{I} \boldsymbol{M}$, we arrive at

$$\boxed{\boldsymbol{I}' = \frac{1}{12}\rho abc \begin{bmatrix} \frac{1}{2}(a^2+b^2)+c^2 & \frac{1}{2}(a^2-b^2) & 0 \\ \frac{1}{2}(a^2-b^2) & \frac{1}{2}(a^2+b^2)+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}} . \tag{18}$$

(c) Consider an *oblate spheroid* of uniform density D, with its principal axes aligned along x, y, z. This is the region that satisfies the inequality

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b} \le 1 , \tag{19}$$

for axes $a > b$ (make $a < b$ if you want it to be prolate). Before calculating anything, make a guess about what the moment of inertia tensor will look like in these coordinates. Now compute its moment of inertia tensor and see if your intuition was correct. Hint: This is probably easiest by writing down the xyz integrals and then changing the integration variables to cylindrical coordinates (z, ρ, ϕ), where as usual $x = \rho\cos\phi$ and $y = \rho\sin\phi$. Of course don't forget the Jacobian factor, and choose the order of integration wisely to make your life as easy as possible.

Solution: As in the previous example, the mass distribution has reflection symmetry across the x, y, and z axes; this implies that the second moment of mass density must be diagonal, and so must the moment of inertia tensor.

Furthermore, since the oblate spheroid is rotationally invariant in the $x-y$ plane, we will have $m_{xx} = m_{yy}$ (isotropic in this subspace).

First, write down the diagonal components of the second moment tensor, in xyz coordinates. We will use $\mathcal{V}$ to denote the interior of this shape.

$$m_{xx} = m_{yy} = D \int_{\mathcal{V}} x^2 d^3x \tag{20}$$

$$m_{zz} = D \int_{\mathcal{V}} z^2 d^3x \tag{21}$$

Now we transform to cylindrical coordinates, and the volume transforms with the determinant of the Jacobian,

$$d^3x = \rho\, d\rho\, d\phi\, dz . \tag{22}$$

Next we must determine the bounds of integration in these coordinates. Since the shape is azimuthally symmetric, $0 \le \phi \le 2\pi$. Clearly the range of z is $-b \le z \le +b$. As for ρ, at each fixed value of z, there is a different maximum ρ, given by the inequality (19). From there we can see that $0 \le \rho \le a\sqrt{1 - z^2/b^2}$.

Transforming the integrals, we have

$$m_{xx} = m_{yy} = D \int_{-b}^{b} \int_0^{2\pi} \int_0^{a\sqrt{1-z^2/b^2}} \rho^2 \cos^2\phi\, \rho\, d\rho\, d\phi\, dz \tag{23}$$

$$m_{zz} = D \int_{-b}^{b} \int_0^{2\pi} \int_0^{a\sqrt{1-z^2/b^2}} z^2\, \rho\, d\rho\, d\phi\, dz \tag{24}$$

First performing the ρ integrals, we only need $\int \rho d\rho$ and $\int \rho^3 d\rho$

$$m_{xx} = m_{yy} = D \int_{-b}^{b} \int_{0}^{2\pi} \cos^2\phi \frac{(a\sqrt{1-z^2/b^2})^4}{4}\, d\phi\, dz = \frac{a^4 D}{4} \int_{-b}^{b} \int_{0}^{2\pi} \cos^2\phi \left(1 - \frac{z^2}{b^2}\right)^2 d\phi\, dz \tag{25}$$

$$m_{zz} = D \int_{-b}^{b} \int_{0}^{2\pi} z^2 \frac{(a\sqrt{1-z^2/b^2})^2}{2}\, d\phi\, dz = \frac{a^2 D}{2} \int_{-b}^{b} \int_{0}^{2\pi} z^2 \left(1 - \frac{z^2}{b^2}\right) d\phi\, dz \tag{26}$$

Next, performing the ϕ integrals, we need $\int_0^{2\pi} \cos^2\phi d\phi = \pi$,

$$m_{xx} = m_{yy} = \frac{\pi a^4 D}{4} \int_{-b}^{b} \left(1 - \frac{z^2}{b^2}\right)^2 dz \tag{27}$$

$$m_{zz} = \pi a^2 D \int_{-b}^{b} z^2 \left(1 - \frac{z^2}{b^2}\right) dz \tag{28}$$

Finally, we have a fourth order polynomial in z in each case that we need to integrate. Performing these integrations yields

$$m_{xx} = m_{yy} = \frac{\pi a^4 D}{4} \frac{16b}{15} \tag{29}$$

$$m_{zz} = \pi a^2 D \frac{4b^3}{15} \tag{30}$$

3. **Surface of a spun-cast mirror**. One way to make a mirror is as follows. Sit a cylindrical vat on a turntable, so that the cylinder is spun around its axis, which is vertical ($\hat{z}$). Let this turntable spin at a frequency ω. Fill this cylinder with molten glass. The rotation of the cylinder couples to the viscous molten glass, making it spin, and it ends up with a curved surface. Now allow the glass to cool slowly, so that it solidifies with the curved surface, which is later given a reflective coating.

   Ignore the rotation of the Earth, and treat gravity as uniform in the $\hat{z}$ direction. Find the parametric form of the surface of the mirror. Hint: once the fluid glass has come into equilibrium, in the rotating frame, none of the fluid elements are moving; what does that mean about the potential difference (which potential?) between different surface fluid elements in the rotating frame?

   Solution: In the lab frame, each infinitesimal fluid element Δm is subject to the potential $V = \Delta m\, gz$. However, the fluid elements are also moving in the lab frame. Meanwhile in the co-rotating frame, once equilibrium is reached, the fluid elements will have no relative velocities.

   If fluid elements are at different potentials, then there would be a force, given by the gradient of the potential, that pushes them toward equal potentials. Therefore once they reach equilibrium, the fluid surface will be $V_{\text{rot}} = \text{const}$ in the rotating frame.

   The rotating potential is the same as the one we discussed in class in the Lagrange point example. There, we found that for a frame rotating with angular frequency ω, the rotating potential on an infinitesimal element Δm would be related to the inertial-frame potential via

$$V_{\text{rot}}(\vec{r}) = V(\vec{r}) - \frac{1}{2}(\Delta m)\omega^2 \vec{\rho}^2\,, \tag{31}$$

   where $\vec{\rho}$ was the component of $\vec{r}$ perpendicular to the rotation axis. Here, since $\vec{\omega} = \omega\hat{z}$, we have $\vec{\rho}^2 = x^2 + y^2$.

   Altogether, this means that the equilibrium fluid surface is given by

$$\text{const} = gz - \frac{1}{2}\omega^2(x^2+y^2) \qquad \Longrightarrow \qquad \boxed{z = \frac{\omega^2}{2g}(x^2+y^2) + c}. \tag{32}$$

   In other words, the fluid surface $z(x, y)$ is parabolic. This is exactly what you want for to focus parallel rays from infinity to a point at the focus of the parabola.

4. **(In)stability of axes in torque-free precession**. Recall that precession is governed by Euler's equations,

$$N_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 \tag{33}$$
$$N_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 \tag{34}$$
$$N_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 \tag{35}$$

which have been evaluated in a body frame which also diagonalizes the moment of inertia tensor (so that $I_{ij} = \text{diag}(I_1, I_2, I_3)$ with $I_1 > I_2 > I_3$), and N_i are the components of external torque in this same body frame. We already saw that when external torques vanish, if the vector $\vec{\omega}$ is aligned with any of the three principal axes, then $\dot{\vec{\omega}} = 0$.

Choose $\vec{\omega}$ along the 1 axis, so $\vec{\omega} = (\omega_1, 0, 0)$. Now suppose we move slightly away from this solution, taking

$$\vec{\omega} = (\omega_1, 0, 0) + \epsilon(\delta\omega_1(t), \delta\omega_2(t), \delta\omega_3(t)) + \mathcal{O}(\epsilon^2)\,. \tag{36}$$

(a) Write out Euler's equations for this ansatz. Neglect terms of $\mathcal{O}(\epsilon^2)$.

Solution: Plugging in this ansatz, the torque-free Euler's equations are

$$0 = \epsilon I_1 \dot{\delta\omega}_1 + \mathcal{O}(\epsilon^2) \tag{37}$$
$$0 = \epsilon I_2 \dot{\delta\omega}_2 + \epsilon(I_1 - I_3)\omega_1\delta\omega_3 + \mathcal{O}(\epsilon^2) \tag{38}$$
$$0 = \epsilon I_3 \dot{\delta\omega}_3 + \epsilon(I_2 - I_1)\delta\omega_2\,\omega_1 + \mathcal{O}(\epsilon^2) \tag{39}$$

(b) You should find that the two equations governing $\delta\omega_2(t)$ and $\delta\omega_3(t)$ are coupled to each other. Take a time derivative of each equation and decouple them.

Solution: Taking a time derivative of the equation that contains $\dot{\delta\omega}_2$ gives us

$$0 = I_2\ddot{\delta\omega}_2 + (I_1 - I_3)\omega_1\dot{\delta\omega}_3 + \mathcal{O}(\epsilon)\,, \tag{40}$$

into which we can insert the equation for $\dot{\delta\omega}_3$. Once we insert this equation, we have

$$\ddot{\delta\omega}_2 = -\frac{(I_1 - I_3)(I_1 - I_2)\omega_1^2}{I_2 I_3}\delta\omega_2\,. \tag{41}$$

The procedure is similar for the $\delta\omega_3$ equation,

$$\ddot{\delta\omega}_3 = -\frac{(I_1 - I_2)(I_1 - I_3)\omega_1^2}{I_2 I_3}\delta\omega_3\,. \tag{42}$$

Let us define

$$\Omega_{23}^2 \equiv \frac{(I_1 - I_2)(I_1 - I_3)\omega_1^2}{I_2 I_3}\,, \tag{43}$$

which appears in both equations. Notice that since $I_1 \geq I_2 \geq I_3$, the quantity Ω_{23}^2 is indeed positive (which justifies calling it the square of some quantity).

(c) What is the general solution for $\delta\omega_{2,3}(t)$?

Solution: Both equations are of the form $\ddot{x} = -\Omega_{23}^2 x$. Therefore the general solution for both $\delta\omega_2(t)$ and $\delta\omega_3(t)$ is

$$\boxed{\delta\omega_{2,3}(t) = A\cos(\Omega_{23}t) + B\sin(\Omega_{23}t)}\,, \tag{44}$$

for some constants A, B which depend on initial conditions. The important quality to note is that both solutions oscillate about zero, staying bounded.

(d) Now repeat supposing we started with

$$\vec{\omega} = (0, \omega_2, 0) + \epsilon(\delta\omega_1(t), \delta\omega_2(t), \delta\omega_3(t)) + \mathcal{O}(\epsilon^2)\,, \tag{45}$$

and again with

$$\vec{\omega} = (0, 0, \omega_3) + \epsilon(\delta\omega_1(t), \delta\omega_2(t), \delta\omega_3(t)) + \mathcal{O}(\epsilon^2)\,. \tag{46}$$

Which axes are stable and which are unstable?

Solution: Rather than redoing the entire derivation, let us note that Euler's equations, when written in index notation (with an ϵ tensor) are invariant under the cyclic permutations $1 \to 2 \to 3 \to 1$ and $1 \to 3 \to 2 \to 1$.

We can get the equations for perturbing about the solution $\vec{\omega} = (0, \omega_2)$ by starting with the ω_1 solution and applying $1 \to 2 \to 3 \to 1$. This gives us the two equations

$$\delta\ddot{\omega}_1 = \frac{(I_1 - I_2)(I_2 - I_3)\omega_2^2}{I_1 I_3}\delta\omega_1 \tag{47}$$

$$\delta\ddot{\omega}_3 = \frac{(I_1 - I_2)(I_2 - I_3)\omega_2^2}{I_1 I_3}\delta\omega_3\,. \tag{48}$$

Here, the factor

$$\gamma_{23}^2 = \frac{(I_1 - I_2)(I_2 - I_3)\omega_2^2}{I_1 I_3} \tag{49}$$

is positive. Instead of a restoring force like in the case of perturbations around the 1 axis, we have a repulsive force. The general solutions are $\delta\omega_{1,3}(t) = A\exp(+\gamma_{23}t) + B\exp(-\gamma_{23}t)$. The positive exponential will dominate over the negative, and the perturbative solution will grow without bound, showing that a perturbative analysis is not valid for long times. Perturbations around the 2 axis are unstable.

Similarly, for perturbations about the 3 axis, we have

$$\delta\ddot{\omega}_1 = -\frac{(I_2 - I_3)(I_1 - I_3)\omega_3^2}{I_1 I_2}\delta\omega_1 \tag{50}$$

$$\delta\ddot{\omega}_2 = -\frac{(I_2 - I_3)(I_1 - I_3)\omega_3^2}{I_1 I_2}\delta\omega_2\,. \tag{51}$$

Here again we have a restoring force and stable oscillations, now with frequency $\sqrt{\frac{(I_2 - I_3)(I_1 - I_3)\omega_3^2}{I_1 I_2}}$.