UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 3 — SOLUTIONS

Due: Monday, Sep. 17, 2018, by 5PM

As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. **Hanging cable**. Consider a cable which can not stretch, so each arc of length ds has a fixed mass density, $dm = \rho ds$. Let the cable be hanging so that its curve is the graph of $z(x)$.

   (a) What is the infinitesimal arc length ds at some point (x, z), in terms of the slope? What is the total length?

   Solution: Consider an infinitesimal unit of length of the arc, and approximate it as the hypotenuse of a right triangle along the (x, z) axes. The horizontal distance is dx and the vertical is $dz = \frac{dz}{dx}dx$.

   Thus the infinitesimal arc length is $\boxed{ds = \sqrt{1 + (\frac{dz}{dx})^2 dx}}$.

   To find the total length, integrate this,

   $$\boxed{L = \int \sqrt{1 + \left(\frac{dz}{dx}\right)^2} dx}\,. \tag{1}$$

   (b) Let the cable be hanging between $-a \leq x \leq +a$, in a uniform gravitational field pointing up along z. What is the total potential energy of the cable?

   Solution: Each infinitesimal element of mass dm contributes a potential energy of $V = gz\,dm$. Thus the total potential energy is

   $$E = \int gzdm = \int gz\rho ds\,, \tag{2}$$

   $$\boxed{E = \int_{-a}^{+a} g\rho z(x) \sqrt{1 + \left(\frac{dz}{dx}\right)^2} dx}\,. \tag{3}$$

   (c) Treat the total potential as an energy functional, and add an appropriate Lagrange multiplier to enforce that the length is constant. Now vary this functional to find a differential equation that $z(x)$ must satisfy.

   Solution: We want to incorporate that the length from Eq. (1) is constant while we vary the energy in Eq. (3). This can be accomplished via the functional

   $$\boxed{F[z(x)] = \int_{-a}^{+a} \left[g\rho z(x) \sqrt{1 + \left(\frac{dz}{dx}\right)^2} + \lambda \left(\sqrt{1 + \left(\frac{dz}{dx}\right)^2} - \text{const.} \right) \right] dx}\,. \tag{4}$$

   Since the constant does not enter into the variation with respect to $z(x)$, it may be dropped. Performing the variation with respect to $z(x)$, much algebra ensues. At first we have

   $$g\rho\sqrt{1 + (z')^2} = \frac{d}{dx}\left[\frac{z'(g\rho z + \lambda)}{\sqrt{1 + (z')^2}} \right]\,. \tag{5}$$

   Taking the derivative and simplifying, we eventually arrive at

   $$\boxed{g\rho(1 + (z')^2) = (g\rho z + \lambda)z''}\,. \tag{6}$$

(d) (Extra credit) Integrate the diffeq so as to find the functional form of a hanging cable.

Solution: This requires a bit of inspiration. Eq. (6) can be rewritten as

$$\frac{2z'z''}{1+(z')^2} = \frac{2g\rho z'}{g\rho z+\lambda}\,. \tag{7}$$

In this form, both sides are total derivatives,

$$\frac{d}{dx}\ln(1+(z')^2) = 2\frac{d}{dz}\ln(g\rho z+\lambda)\,. \tag{8}$$

Integrate (don't forget the integration constant) and then exponentiate,

$$1+(z')^2 = C(g\rho z+\lambda)^2\,. \tag{9}$$

We can fix the integration constant by moving the origin of coordinates. Find the minimum of the hanging cable between the two endpoints, and call this $x=0, z=0$. Since it is a minimum, $z'=0$ as well. This means that $C = 1/\lambda^2$.

Solve for z',

$$\frac{dz}{dx} = \sqrt{\frac{1}{\lambda^2}(g\rho z+\lambda)^2 - 1} \qquad \Longrightarrow \qquad dx = \frac{\lambda dz}{\sqrt{(g\rho z+\lambda)^2-\lambda^2}} \tag{10}$$

Integrate Eq. (10) up from $x=0, z=0$ to $(x, z(x))$. The RHS can be simplified by changing variables via $u = g\rho z/\lambda$, giving

$$x = \frac{\lambda}{g\rho}\int_0^{g\rho z(x)/\lambda} \frac{du}{\sqrt{u(2+u)}}\,. \tag{11}$$

This is not an elementary integral, but substituting $u = 2k^2$, $du = 4kdk$ will do the trick:

$$x = \frac{2\lambda}{g\rho}\int_0^{\sqrt{g\rho z(x)/2\lambda}} \frac{dk}{\sqrt{1+k^2}} = \frac{2\lambda}{g\rho}\,\mathrm{arcsinh}\sqrt{\frac{g\rho z}{2\lambda}}\,. \tag{12}$$

Solving for $z(x)$ then gives

$$z(x) = \frac{2\lambda}{g\rho}\sinh^2\frac{g\rho x}{2\lambda} = \boxed{\frac{\lambda}{g\rho}\left[\cosh\left(\frac{g\rho x}{\lambda}\right)-1\right]}\,. \tag{13}$$

With this explicit form for $z(x)$, we can then integrate Eq. (1) to find a transcendental equation between L and g, ρ, and λ. This allows to (implicitly) solve for λ.

2. **Justifying your geometric intuition**. Take a cable again of length $L > 2a$ fixed in the x–y plane at $(-a, 0)$ and $(+a, 0)$. Consider the area bounded by the cable and the x axis, see Fig. 1. Write down an appropriate functional with Lagrange multiplier in order to maximize this area, subject to the constraint that the length must be constant. What does your intuition say the shape should be? Does the solution agree?

Solution: Intuition suggests that the cable should follow a segment of a circle.

Let the cable follow the curve described by the graph of the function $y(x)$. Then the area bounded by the curve is

$$A = \int_{-a}^{a} y dx\,. \tag{14}$$

We want to vary this subject to the constraint that the length is fixed. As in the previous problem, the length is

$$L = \int_{-a}^{+a} \sqrt{1+y'^2}dx\,. \tag{15}$$

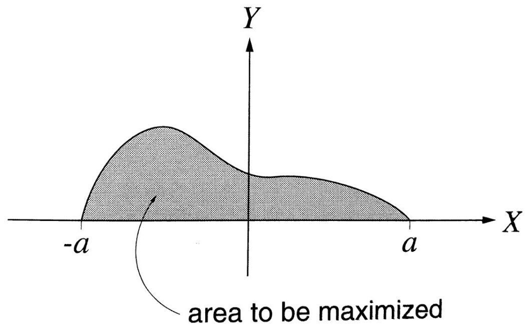

Figure 1: Area to be maximized

Therefore we will use the functional

$$F[y(x)] = \int_{-a}^{+a} \left(y + \lambda\sqrt{1+y'^2}\right) dx\,. \tag{16}$$

Variation with respect to y gives the differential equation

$$1 = \frac{d}{dx}\frac{\lambda y'}{\sqrt{1+y'^2}}\,. \tag{17}$$

This can be immediately integrated with respect to x, with a constant, to give

$$x + c = \frac{\lambda y'}{\sqrt{1+y'^2}}\,. \tag{18}$$

By symmetry, the curve must be symmetric under the reflection $x \to -x$; thus the slope at $x = 0$ must vanish, $y'(0) = 0$. Therefore we find $c = 0$.

Solving for y', we then have to perform another integration,

$$y'^2 = \frac{x^2}{\lambda^2 - x^2} \qquad \Longrightarrow \qquad dy = \frac{\pm x dx}{\sqrt{\lambda^2 - x^2}}\,. \tag{19}$$

This can be turned into an elementary integral with the substitution $u = x^2$. Integrate along with the boundary condition that $y(\pm a) = 0$ to find

$$\boxed{y = \sqrt{\lambda^2 - x^2} - \sqrt{\lambda^2 - a^2}}\,. \tag{20}$$

Since we have $(y + \sqrt{\lambda^2 - a^2})^2 + x^2 = \lambda^2$, this is indeed a segment of a circle, with radius λ. The center is at $(x, y) = (0, -\sqrt{\lambda^2 - a^2})$. By integrating the arc length, we may find λ in terms of a and L. Without showing the full derivation, the result is $L = 2\lambda \arcsin(a/\lambda)$, so λ is only known as the root of a transcendental equation.

3. **Properties of orthogonal matrices**. Recall that an $n \times n$ matrix $\boldsymbol{U}$ is called orthogonal if it satisfies $\boldsymbol{U}^T\boldsymbol{U} = \mathbf{1}$, the identity matrix.

   (a) Prove that $\det(\boldsymbol{U}) = \pm 1$ (a rotation with positive determinant is called "proper," and with negative determinant "improper"; but that is just a composition of a proper rotation and a reflection).
   Solution: Take the determinant of the defining relation,

$$\det(\boldsymbol{U}^T\boldsymbol{U}) = \det(\mathbf{1}) \tag{21}$$

$$\det(\boldsymbol{U}^T)\det(\boldsymbol{U}) = 1 \tag{22}$$

$$\det(\boldsymbol{U})\det(\boldsymbol{U}) = 1 \tag{23}$$

where we evaluated the determinant of the identity matrix is 1, and the determinant of the product of square matrices is the product of their determinants. Next, we use the identity that a matrix and its transpose have the same determinant. This gives us the fact that $\det(\boldsymbol{U})^2 = 1$, or taking the square root, $\det(\boldsymbol{U}) = \pm 1$.

(b) For a proper rotation in any odd dimension n, prove that the orthogonal matrix $\boldsymbol{U}$ has at least one eigenvalue equal to 1; hence there is an "axis" of rotation – a direction that is invariant under the transformation $\boldsymbol{U}$. (Hint: first prove that if there is an eigenvalue equal to 1, then $\det(\boldsymbol{U} - \mathbf{1}) = 0$.)

Solution: The eigenvalues of $\boldsymbol{U}$ are roots $p(\lambda) = 0$ of the characteristic polynomial $p(x) = \det(\boldsymbol{U} - x\mathbf{1})$. Thus, if 1 is an eigenvalue, $p(1) = 0 = \det(\boldsymbol{U} - \mathbf{1})$.

Now, consider the product

$$\boldsymbol{U}^T(\boldsymbol{U} - \mathbf{1}) = \mathbf{1} - \boldsymbol{U}^T = -(\boldsymbol{U} - \mathbf{1})^T \,. \tag{24}$$

Taking the determinant of both sides we find

$$\det(\boldsymbol{U}^T)\det(\boldsymbol{U} - \mathbf{1}) = \det(-(\boldsymbol{U} - \mathbf{1})^T) \tag{25}$$

$$\det(\boldsymbol{U} - \mathbf{1}) = (-1)^n \det(\boldsymbol{U} - \mathbf{1}) \,, \tag{26}$$

from the facts that $\det \boldsymbol{M} = \det(\boldsymbol{M}^T)$; $\det \boldsymbol{U} = 1$ because $\boldsymbol{U}$ is a proper rotation; and that the determinant of a scalar k times an $n \times n$ matrix $\boldsymbol{M}$ is $\det(k\boldsymbol{M}) = k^n \det \boldsymbol{M}$.

Now in Eq. (26), when n is even, we have the uninformative equation $\det(\boldsymbol{U} - \mathbf{1}) = \det(\boldsymbol{U} - \mathbf{1})$. However, when n is odd, we find $\det(\boldsymbol{U} - \mathbf{1}) = -\det(\boldsymbol{U} - \mathbf{1})$. This can only be satisfied if $\det(\boldsymbol{U} - \mathbf{1}) = 0$. Therefore when n is odd, we always have an eigenvalue of 1.

(c) The trace of a matrix $\boldsymbol{A}$ is $\mathrm{tr}(\boldsymbol{A}) = \sum_i A_{ii}$, the sum of the diagonal elements. Prove that the trace is invariant under a similarity transformation by an orthogonal matrix, i.e. that if $\boldsymbol{A}' = \boldsymbol{U}^T\boldsymbol{A}\boldsymbol{U}$, then $\mathrm{tr}(\boldsymbol{A}') = \mathrm{tr}(\boldsymbol{A})$. (Hint: first prove that $\mathrm{tr}(\boldsymbol{AB}) = \mathrm{tr}(\boldsymbol{BA})$.)

Solution: Writing out the trace of a product,

$$\mathrm{tr}(\boldsymbol{AB}) = \sum_i (\boldsymbol{AB})_{ii} = \sum_i \sum_j A_{ij}B_{ji} = \sum_{ij} B_{ij}A_{ji} \tag{27}$$

where the last equality is just replacing the dummy indices $i \leftrightarrow j$. Therefore we have $\mathrm{tr}(\boldsymbol{AB}) = \mathrm{tr}(\boldsymbol{BA})$.

Note that we have to be careful when taking the trace of a product of 3 or more matrices—you have to put parentheses in to make it the product of two matrices, e.g.

$$\mathrm{tr}(\boldsymbol{ABC}) = \mathrm{tr}(\boldsymbol{A}(\boldsymbol{BC})) = \mathrm{tr}((\boldsymbol{BC})\boldsymbol{A}) \,. \tag{28}$$

You can not reorder the matrix products any way you like—you can only move one matrix at a time from the beginning to the end or vice versa. Generalizing to more matrices, we have $\mathrm{tr}(\boldsymbol{M}_1\boldsymbol{M}_2\cdots\boldsymbol{M}_k) = \mathrm{tr}(\boldsymbol{M}_k\boldsymbol{M}_1\boldsymbol{M}_2\cdots\boldsymbol{M}_{k-1}) = \mathrm{tr}(\boldsymbol{M}_2\cdots\boldsymbol{M}_k\boldsymbol{M}_1)$. We say that the trace is *cyclic*.

Now take the trace of the similarity transformation,

$$\boldsymbol{A}' = \boldsymbol{U}^T\boldsymbol{A}\boldsymbol{U} \tag{29}$$

$$\mathrm{tr}(\boldsymbol{A}') = \mathrm{tr}(\boldsymbol{U}^T\boldsymbol{A}\boldsymbol{U}) \tag{30}$$

$$\boldsymbol{A}' = \mathrm{tr}(\boldsymbol{U}\boldsymbol{U}^T\boldsymbol{A}) = \mathrm{tr}(\boldsymbol{A}) \tag{31}$$

using the cyclicity of the trace, and then by definition of an orthogonal matrix, $\boldsymbol{U}\boldsymbol{U}^T = \mathbf{1}$.

(d) **Special case of** 3×3. If $\mathbf{U}$ is a 3×3 orthogonal matrix, show that its trace is $\mathrm{tr}(\boldsymbol{U}) = 1 + 2\cos\phi$, where ϕ is the angle of rotation.

Solution: Since the trace is invariant under a similarity transformation, instead of evaluating the trace of a general orthogonal matrix, we can perform a similarity transformation to align the axis of rotation with the z axis. Then we only need to evaluate the trace of a rotation about the z axis,

$$\mathrm{tr}(\boldsymbol{R}_z(\phi)) = \mathrm{tr}\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix} = 1 + 2\cos\phi \,. \tag{32}$$

4. **What is this matrix**? Consider the matrix A with components

$$A_{jk} = (1 - \cos\phi)\hat{n}_j\hat{n}_k + \cos\phi\,\delta_{jk} - \sin\phi\,\hat{n}_i\epsilon_{ijk} \tag{33}$$

with $\hat{n}_i$ the components of a unit vector $\hat{n}$, ϵ_{ijk} the Levi-Civita symbol, and I am using the Einstein summation convention for repeated indices. The following identities are useful.

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{34}$$
$$\epsilon_{ijk}\hat{n}_j\hat{n}_k = 0. \tag{35}$$

(a) Show that A is an orthogonal matrix, and so can be considered to be a rotation matrix.
Solution: Let's start by multiplying out $\boldsymbol{AA}^T$, where $(\boldsymbol{A}^T)_{ij} = A_{ji}$. Writing out the product gives

$$\begin{aligned}
(\boldsymbol{AA}^T)_{ij} &= A_{ik}A^T_{kj} = A_{ik}A_{jk} && (36)\\
&= [(1-\cos\phi)\hat{n}_i\hat{n}_k + \cos\phi\,\delta_{ik} - \sin\phi\,\hat{n}_l\epsilon_{lik}]\times\\
&\qquad [(1-\cos\phi)\hat{n}_j\hat{n}_k + \cos\phi\,\delta_{jk} - \sin\phi\,\hat{n}_m\epsilon_{mjk}] && (37)\\
&= (1-\cos\phi)^2\hat{n}_i\hat{n}_k\hat{n}_j\hat{n}_k + (1-\cos\phi)\cos\phi\,\hat{n}_i\hat{n}_k\delta_{jk}\\
&\qquad - (1-\cos\phi)\sin\phi\,\underline{\hat{n}_i\hat{n}_k\hat{n}_m\epsilon_{mjk}} + \cos\phi(1-\cos\phi)\,\delta_{ik}\hat{n}_j\hat{n}_k\\
&\qquad + \cos^2\phi\,\delta_{ik}\delta_{jk} - \cos\phi\sin\phi\,\delta_{ik}\hat{n}_m\epsilon_{mjk}\\
&\qquad - \sin\phi(1-\cos\phi)\,\underline{\hat{n}_l\epsilon_{lik}\hat{n}_j\hat{n}_k} - \sin\phi\cos\phi\,\hat{n}_l\epsilon_{lik}\delta_{jk}\\
&\qquad + \sin^2\phi\,\hat{n}_l\epsilon_{lik}\,\hat{n}_m\epsilon_{mjk}\,, && (38)
\end{aligned}$$

where we have underlined the terms containing $\hat{n}\times\hat{n} = \epsilon_{ijk}\hat{n}_j\hat{n}_k = 0$. Throw out those terms, use $\hat{n}\cdot\hat{n} = \hat{n}_i\hat{n}_i = 1$, contract some δ symbols, and use the identity for the product of two ϵ's to get

$$\begin{aligned}
(\boldsymbol{AA}^T)_{ij} &= (1-\cos\phi)^2\hat{n}_i\hat{n}_j + (1-\cos\phi)\cos\phi\,\hat{n}_i\hat{n}_j\\
&\qquad + \cos\phi(1-\cos\phi)\,\hat{n}_j\hat{n}_i\\
&\qquad + \cos^2\phi\,\delta_{ij} - \underline{\cos\phi\sin\phi\,\hat{n}_m\epsilon_{mji}}\\
&\qquad - \underline{\sin\phi\cos\phi\,\hat{n}_l\epsilon_{lij}}\\
&\qquad + \sin^2\phi\,\hat{n}_l\hat{n}_m(\delta_{lm}\delta_{ij} - \delta_{lj}\delta_{im})\,. && (39)
\end{aligned}$$

Note that the two newly underlined terms are the same except for a different dummy index, and an exchange of $i \leftrightarrow j$; but this exchange flips the sign of the ϵ, so these two cancel. Continuing, expand out the parentheses in the last line and contract the δ's,

$$\begin{aligned}
(\boldsymbol{AA}^T)_{ij} &= (1-\cos\phi)^2\hat{n}_i\hat{n}_j + (1-\cos\phi)\cos\phi\,\hat{n}_i\hat{n}_j\\
&\qquad + \cos\phi(1-\cos\phi)\,\hat{n}_j\hat{n}_i\\
&\qquad + \cos^2\phi\,\delta_{ij}\\
&\qquad + \sin^2\phi\,\hat{n}_l\hat{n}_m\delta_{lm}\delta_{ij} - \sin^2\phi\,\hat{n}_l\hat{n}_m\delta_{lj}\delta_{im} && (40)\\
(\boldsymbol{AA}^T)_{ij} &= (1-\cos\phi)^2\hat{n}_i\hat{n}_j + (1-\cos\phi)\cos\phi\,\hat{n}_i\hat{n}_j\\
&\qquad + \cos\phi(1-\cos\phi)\,\hat{n}_j\hat{n}_i\\
&\qquad + \cos^2\phi\,\delta_{ij}\\
&\qquad + \sin^2\phi\,\delta_{ij} - \sin^2\phi\,\hat{n}_i\hat{n}_j\,. && (41)
\end{aligned}$$

All the terms we are left with are proportional to either δ_{ij} or $\hat{n}_i\hat{n}_j$. Collecting,

$$(\boldsymbol{AA}^T)_{ij} = \delta_{ij}[\cos^2\phi + \sin^2\phi] + \hat{n}_i\hat{n}_j[(1-\cos\phi)^2 + 2(1-\cos\phi)\cos\phi - \sin^2\phi] \tag{42}$$
$$(\boldsymbol{AA}^T)_{ij} = \delta_{ij}\,. \tag{43}$$

Thus we have shown that $\boxed{\boldsymbol{AA}^T = \mathbf{1}}$, the defining property of an orthogonal matrix.

(b) Write down the matrices A for (i) $\hat{n} = \hat{x}$, (ii) $\hat{n} = \hat{y}$, and (iii) $\hat{n} = \hat{z}$.

Solution:

i. For $\hat{n} = \hat{x}$, we have $\hat{n}_1 = 1, \hat{n}_2 = 0, \hat{n}_3 = 0$. This makes

$$\hat{n}_j \hat{n}_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{44}$$

where j is labeling the rows and k is labeling the columns, and

$$\hat{n}_i \epsilon_{ijk} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix} . \tag{45}$$

Plugging these expressions (and δ which gives the identity matrix) into the definition in Eq. (33), we get

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & +\sin\phi & \cos\phi \end{pmatrix} . \tag{46}$$

ii. Now take $\hat{n} = \hat{y}$ which gives $\hat{n}_1 = 0, \hat{n}_2 = 1, \hat{n}_3 = 0$, and proceeding as above,

$$\boldsymbol{A} = \begin{pmatrix} \cos\phi & 0 & +\sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} . \tag{47}$$

iii. Now take $\hat{n} = \hat{z}$ which gives $\hat{n}_1 = 0, \hat{n}_2 = 0, \hat{n}_3 = 1$, and proceeding as above,

$$\boldsymbol{A} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ +\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{48}$$

(c) Show that $\mathrm{tr}(\boldsymbol{A}) = 1 + 2\cos\phi$.

Solution: Writing out the contraction,

$$\mathrm{tr}(\boldsymbol{A}) = A_{ii} = (1 - \cos\phi)\hat{n}_i\hat{n}_i + \cos\phi\,\delta_{ii} - \sin\phi\,\hat{n}_l\epsilon_{lii} \tag{49}$$

$$\mathrm{tr}(\boldsymbol{A}) = (1 - \cos\phi) + 3\cos\phi - 0 = 1 + 2\cos\phi\,. \tag{50}$$

Here we have used that $\delta_{ii} = 3$, being the trace of the 3×3 identity matrix, and $\epsilon_{iij} = 0$ because the ϵ tensor is antisymmetric on any pair of indices; so if you make two indices the same, and then exchange them, you get the negative, so it must be 0.

(d) Show that a vector parallel to $\hat{n}$ is *not* changed by the rotation corresponding to A.

Solution: The transformation is $\mathbb{R}$-linear, so it suffices to show that $\boldsymbol{A}\hat{n} = \hat{n}$. Writing this out in components,

$$(\boldsymbol{A}\hat{n})_j = A_{jk}\hat{n}_k = (1 - \cos\phi)\hat{n}_j\hat{n}_k\hat{n}_k + \cos\phi\,\delta_{jk}\hat{n}_k - \sin\phi\,\hat{n}_i\epsilon_{ijk}\hat{n}_k \tag{51}$$

$$= (1 - \cos\phi)\hat{n}_j + \cos\phi\,\hat{n}_j - 0 \tag{52}$$

where the last term has cancelled from the identity $\epsilon_{ijk}\hat{n}_j\hat{n}_k = 0$. The two remaining terms combine to give $\boldsymbol{A}\hat{n} = \hat{n}$, and thus any vector proportional to $\hat{n}$ is unchanged.

(e) From parts (a)-(d) we can argue that A_{jk} is the rotation matrix for a rotation with axis $\hat{n}$ and rotation angle $\pm\phi$. Make some simple choice for $\hat{n}$ to determine whether the plus or minus sign should be used.

Solution: Examining the case $\hat{n} = \hat{z}$ from part (b), we can see that $\boldsymbol{A}$ has the standard sign convention for right-handed rotations.