UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 2 — SOLUTIONS

Due: Friday, Sep. 7, 2018, by 5PM

I'm attempting to perform a binary search in the space of amount of homework to assign, so there are far fewer problems this week than last week. As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. FW1.11

Solution: As usual, we have an effective potential, in this case given by

$$V_{\text{eff}} = \frac{\ell^2}{2\mu r^2} - \frac{\lambda}{r^n}\,, \tag{1}$$

where recall that we are interested in $\lambda > 0, n > 2$. As $r \to \infty$, the positive and repulsive angular momentum barrier (the ℓ^2 bit) dominates. As $r \to 0$, the negative and attractive potential (the λ bit) dominates. The case $\ell = 0$ is a plunge with no angular momentum barrier. All other cases $\ell \neq 0$ have a maximum in the effective potential at some point r_{peak} which solves the equation $V'_{\text{eff}}(r_{\text{peak}}) = 0$. There are just two terms,

$$V'_{\text{eff}} = \frac{-\ell^2}{\mu r^3} + \frac{n\lambda}{r^{n+1}}\,. \tag{2}$$

Setting this equal to 0 gives $r_{\text{peak}} = |n\lambda\mu/\ell^2|^{1/(n-2)}$.

At an energy of $E_{\text{peak}} = V(r_{\text{peak}})$, there is an unstable circular orbit. For any energy $E < E_{\text{peak}}$, there are two types of orbits: i) a scattering orbit at radii exterior to r_{peak}; and ii) a bound orbit at radii interior to r_{peak}.

Suppose that at t_0, a particle on a bound orbit (interior to the peak) starts at a finite radius r_0. We want to determine both the time and the number of orbits for the particle to reach $r = 0$. It's clear that the particle will reach the origin. The time comes from the implicit solution to the radial equation of motion,

$$t - t_0 = \sqrt{\frac{\mu}{2}} \int_{r_0}^{r(t)} \frac{dr}{\pm\sqrt{E - V_{\text{eff}}(r)}}\,. \tag{3}$$

The plus/minus sign here depends on whether the particle is going to the left or right (the time must monotonically increase). Let's suppose the particle is always going to decreasing r, and thus take the minus sign.

All we need to show is that the integral in Eq. (3) is finite for $r(t) = 0$. Let us split the radial interval into two pieces, one exterior to some small (but finite) value $\epsilon > 0$, and one interior. Clearly the exterior integral is finite. The only danger is if the interior integral diverges. For a sufficiently small $r < \epsilon$, $\lambda r^{-n} \gg |E|$. Then the value of the integrand can be approximated as

$$\frac{1}{\sqrt{E - V_{\text{eff}}(r)}} \approx \frac{r^{n/2}}{\sqrt{\lambda}}\,. \tag{4}$$

Thus the contribution of the integral for $0 \leq r \leq \epsilon$ will be

$$t + \text{const} \sim \sqrt{\frac{\mu}{2\lambda}} \int_0^{\epsilon} r^{n/2} dr = \sqrt{\frac{\mu}{2\lambda}} \left[\frac{r^{n/2+1}}{n/2+1}\right]_0^{\epsilon}\,, \tag{5}$$

which is clearly a finite number, since $n/2+1>0$. Therefore the particle reaches the origin in a finite amount of time.

The question of the number of orbits is similar. Recall that the angular equation can be integrated to give

$$\phi - \phi_0 = \frac{\ell}{\sqrt{2\mu}} \int_{r_0}^{r(t)} \frac{dr}{\pm r^2 \sqrt{E - V_{\text{eff}}(r)}} \,. \tag{6}$$

Now we can see that this integral is a bit more dangerous, since there is an additional r^{-2} in the integrand. Doing the same analysis as above, for $0 \le r \le \epsilon$, the contribution to the azimuthal phase is

$$\phi + \text{const} \sim \frac{\ell}{\sqrt{2\mu\lambda}} \int_0^{\epsilon} r^{n/2-2} dr = \frac{\ell}{\sqrt{2\mu\lambda}} \left[\frac{r^{n/2-1}}{n/2-1} \right]_0^{\epsilon} . \tag{7}$$

Once again, since $n>2$, $n/2-1>0$, so the amount of phase accrued also converges to a finite number.

Since the above quantities are finite, it is conceivable that one can integrate through the infinite force at the origin. In that case, since the particle will pass through the origin, the radial motion will flip from ingoing to outgoing, so $\dot{r}$ will flip signs (keeping the same magnitude, so as to conserve E). Meanwhile, since the particle passes straight through, the azimuthal phase will jump by a factor of π. This is easiest to see by examining the motion in the x–y plane.

2. **Angular momentum under Galilean invariance**. For this problem, we will consider a system of N particles in 3 dimensions, with no external forces. Let the Lagrangian be of the form

$$L = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{\mathbf{r}}}_i^2 - \sum_{i \neq j} V_{ij}(r_{ij}) \,, \tag{8}$$

where recall that $\vec{\mathbf{r}}_{ij} \equiv \vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j$, and $r_{ij} = |\vec{\mathbf{r}}_{ij}| = \sqrt{\vec{\mathbf{r}}_{ij} \cdot \vec{\mathbf{r}}_{ij}}$.

(a) Find the variational derivatives

$$\frac{\delta r_{ij}}{\delta \vec{\mathbf{r}}_i} \,, \quad \frac{\delta r_{ij}}{\delta \vec{\mathbf{r}}_j} \,, \qquad \text{and} \qquad \frac{\delta r_{ij}}{\delta \vec{\mathbf{r}}_k} \quad (k \neq i \,, k \neq j) \,. \tag{9}$$

Recall that the variational derivative δ acts like a "derivation" and thus satisfies a chain rule and product rule.

Solution: Applying the chain rule, find:

$$\frac{\delta}{\delta \vec{\mathbf{r}}_i} r_{ij} = \frac{\delta}{\delta \vec{\mathbf{r}}_i} \sqrt{\vec{\mathbf{r}}_{ij} \cdot \vec{\mathbf{r}}_{ij}} = \frac{1}{2\sqrt{\vec{\mathbf{r}}_{ij} \cdot \vec{\mathbf{r}}_{ij}}} 2 \vec{\mathbf{r}}_{ij} \cdot \frac{\delta}{\delta \vec{\mathbf{r}}_i} \vec{\mathbf{r}}_{ij} \,. \tag{10}$$

Clearly we need to find $\delta \vec{\mathbf{r}}_{ij} / \delta \vec{\mathbf{r}}_i$. The degrees of freedom of the ith and jth particles are independent because they do not have any constraints between them. Therefore

$$\frac{\delta}{\delta \vec{\mathbf{r}}_i} \vec{\mathbf{r}}_{ij} = \frac{\delta}{\delta \vec{\mathbf{r}}_i} \vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j = \frac{\delta}{\delta \vec{\mathbf{r}}_i} \vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j = \frac{\delta}{\delta \vec{\mathbf{r}}_i} \vec{\mathbf{r}}_i \,. \tag{11}$$

Now, this is in fact shorthand for a *matrix* of derivatives, since both the numerator and denominator are vector-valued. In full, this is

$$\frac{\delta}{\delta \vec{\mathbf{r}}_i} \vec{\mathbf{r}}_i = \frac{\delta(x_i, y_i, z_i)}{\delta(x_i, y_i, z_i)} = \begin{bmatrix} \frac{\delta x_i}{\delta x_i} & \frac{\delta y_i}{\delta x_i} & \frac{\delta z_i}{\delta x_i} \\ \frac{\delta x_i}{\delta y_i} & \frac{\delta y_i}{\delta y_i} & \frac{\delta z_i}{\delta y_i} \\ \frac{\delta x_i}{\delta z_i} & \frac{\delta y_i}{\delta z_i} & \frac{\delta z_i}{\delta z_i} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{12}$$

that is, the identity matrix.

Putting this together, we have

$$\boxed{\frac{\delta}{\delta \vec{\mathbf{r}}_i} r_{ij} = \frac{\vec{\mathbf{r}}_{ij}}{r_{ij}} = \hat{\vec{\mathbf{r}}}_{ij}}, \tag{13}$$

the unit vector pointing from particle j to particle i.

The only difference when computing $\delta r_{ij}/\delta \vec{\mathbf{r}}_j$ is in finding $\delta \vec{\mathbf{r}}_{ij}/\delta \vec{\mathbf{r}}_j = -\mathbb{I}$. Therefore you find

$$\boxed{\frac{\delta}{\delta \vec{\mathbf{r}}_j} r_{ij} = -\frac{\vec{\mathbf{r}}_{ij}}{r_{ij}} = -\hat{\vec{\mathbf{r}}}_{ij} = +\hat{\vec{\mathbf{r}}}_{ji}}. \tag{14}$$

Of course, $\boxed{\delta r_{ij}/\delta \vec{\mathbf{r}}_k = 0}$ when $k \neq i$, $k \neq j$.

(b) What are the equations of motion for the ith particle? What can you note about the force on the ith particle due to the jth particle, as compared to the force on the jth particle due to the ith particle?

Solution: Varying the Lagrangian in Eq. (8) with respect to $\vec{\mathbf{r}}_i$, we find the vector equation of motion

$$\boxed{m_i \ddot{\vec{\mathbf{r}}}_i = -\sum_{j \neq i} V'_{ij}(r_{ij}) \hat{\vec{\mathbf{r}}}_{ij}}. \tag{15}$$

(c) What is the total angular momentum of the mechanical system, about the origin?

Solution: The total angular momentum is given by

$$\boxed{\vec{\mathbf{L}} \equiv \sum_{i=1}^{N} \vec{\mathbf{r}}_i \times \vec{\mathbf{p}}_i = \sum_{i=1}^{N} \vec{\mathbf{r}}_i \times \vec{\mathbf{p}}_i = \sum_{i=1}^{N} m_i \vec{\mathbf{r}}_i \times \dot{\vec{\mathbf{r}}}_i}. \tag{16}$$

(d) What is the time derivative of the total angular momentum?

Solution: Take the time derivative of Eq. (16) and use the product rule to find

$$\frac{d}{dt} \vec{\mathbf{L}} = \sum_{i=1}^{N} m_i \left[\dot{\vec{\mathbf{r}}}_i \times \dot{\vec{\mathbf{r}}}_i + \vec{\mathbf{r}}_i \times \ddot{\vec{\mathbf{r}}}_i \right] . \tag{17}$$

The first term inside the sum vanishes by the antisymmetry of the cross product. To evaluate the second term, we need to insert the equations of motion, Eq. (15). When we plug this in we get (notice that the m_i cancels!)

$$\frac{d}{dt} \vec{\mathbf{L}} = \sum_{i=1}^{N} \sum_{j \neq i} -V'_{ij}(r_{ij}) \vec{\mathbf{r}}_i \times \hat{\vec{\mathbf{r}}}_{ij} . \tag{18}$$

Now to evaluate the cross product,

$$\vec{\mathbf{r}}_i \times \hat{\vec{\mathbf{r}}}_{ij} = \vec{\mathbf{r}}_i \times \frac{\vec{\mathbf{r}}_{ij}}{r_{ij}} = \vec{\mathbf{r}}_i \times \frac{-\vec{\mathbf{r}}_j}{r_{ij}} , \tag{19}$$

by expanding $\vec{\mathbf{r}}_{ij} = \vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j$ and again using that $\vec{v} \times \vec{v} = 0$ for any vector $\vec{v}$ by the antisymmetry of the cross product. Thus we have

$$\frac{d}{dt} \vec{\mathbf{L}} = \sum_{i=1}^{N} \sum_{j \neq i} -\frac{V'_{ij}(r_{ij})}{r_{ij}} \vec{\mathbf{r}}_i \times \vec{\mathbf{r}}_j . \tag{20}$$

Now consider any pair of (i, j) with $i \neq j$. This pair contributes two terms in the sum (the scalar prefactor is the same in both terms),

$$-\frac{V'_{ij}(r_{ij})}{r_{ij}} \left(\vec{\mathbf{r}}_i \times \vec{\mathbf{r}}_j + \vec{\mathbf{r}}_j \times \vec{\mathbf{r}}_i\right) . \tag{21}$$

But again from the antisymmetry of the cross product, these two summands cancel out. In words, the torque from particle i on particle j and the torque from particle j on particle i make equal and opposite contributions to the time derivative of the total angular momentum.

Thus we have shown $\boxed{d\vec{\mathbf{L}}/dt = 0}$.

3. **Order of solving/varying**. Let's look at a 2-body central force problem, now already reduced to spherical polar coordinates, with the Lagrangian being

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - V(r)\,. \tag{22}$$

Recall that the p_ϕ momentum, traditionally called ℓ, is constant,

$$\mu r^2 \sin^2\theta\,\dot{\phi} = \ell = \text{const.} \tag{23}$$

(a) What is the conserved energy E associated with the Lagrangian in Eq. (22)?
Solution: The conserved energy is

$$\boxed{E = T + V = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) + V(r)}\,. \tag{24}$$

(b) Eliminate $\dot{\phi}$ from E by using the conserved ℓ from Eq. (23). Now identify the effective potential $V_{\text{eff}}(r)$ (assume that the particle orbits in the x–y plane).
Solution: If the particle starts in the orbital plane and stays there, then $\theta = \pi/2$ and $\dot{\theta} = 0$. Now we solve for $\dot{\phi}$ in Eq. (23) and plug all this into the conserved energy, we find

$$E = \frac{\mu}{2}\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)\,. \tag{25}$$

If this is to be the energy for an effective 1-DoF problem, it should look like $E = \frac{\mu}{2}\dot{r}^2 + V_{\text{eff}}(r)$. By identification we then find that $\boxed{V_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} + V(r)}$.

(c) Now try the opposite order. Try to eliminate $\dot{\phi}$ from L by using the conserved ℓ from Eq. (23). Does this give the same effective potential as before?
Solution: Now we try plugging in $\dot{\phi}$ from Eq. (23) into the Lagrangian, Eq. (22). This gives

$$L \stackrel{?}{=} \frac{\mu}{2}\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - V(r)\,. \tag{26}$$

If this was the Lagrangian for an effective 1-DoF problem it would look like $L = \frac{\mu}{2}\dot{r}^2 - V_{\text{eff}}(r)$. Thus by identification we would find that $\boxed{V_{\text{eff}}(r) \stackrel{?}{=} \frac{-\ell^2}{2\mu r^2} + V(r)}$. Notice that the sign of the angular momentum barrier is opposite of what we found before!

(d) Can you explain why? (Hint: variation of the action must be performed by letting the generalized coordinates vary independently)
Solution: The original Lagrangian in Eq. (22) gives correct equations of motion when varied with respect to $\delta r, \delta\phi, \delta\theta$, which all vary independently of each other. If we want to eliminate ϕ using the constraint Eq. (23), then this constraint tells us that the variations of r, θ, ϕ will no longer be

independent. Specifically, taking the variation of Eq. (23), we find that the variations would be constrained via

$$\mu\left(2r\delta r\sin^2\theta\dot{\phi} + r^2 2\sin\theta\cos\theta\delta\theta\dot{\phi} + r^2\sin^2\theta\dot{\delta\phi}\right) = 0\,. \tag{27}$$

It is clear that enforcing this constraint would mean we can not independently vary δr in Eq. (26), and thus it can not act as an ordinary Lagrangian for a 1-DoF problem.

We are not using the conserved energy E in Eq. (25) as some sort of functional to determine the equations of motion; it is just a number, and plugging in the value of ℓ is perfectly valid.

One can always check with the original equations of motion from the full L in Eq. (22) to see which method is correct. Here we find that plugging into the energy was valid, so part (b) was correct; and plugging into the Lagrangian was wrong, as stated before, because we can't vary independently.

4. **Computer tool practice** (not for credit). Computers are crucial research tools for physics nowadays. This exercises are not for credit because this is not a programming or numerical methods course, and since everyone has a different amount of experience with programming and numerics, it would be unfair to grade this work for credit. However I still encourage everyone to try this exercise.

   Extra feel-good points (not redeemable for credit) for making your code easy to understand, well documented, or even using a revision control system (e.g. git) to keep track of your code.

   (a) Using Mathematica, python, or your favorite programming environment: Generate a contour plot of energy in the θ–$\dot{\theta}$ plane for the rigid pendulum, whose Lagrangian (not energy!) is

   $$L = \frac{1}{2}\dot{\theta}^2 - \cos\theta\,. \tag{28}$$

   Here we have adjusted the units so that $mL^2 = 1$ and $mgL = 1$ (convince yourself that this is possible).

   Make sure that the following features are visible:

   i. The periodicity in θ
   ii. The separatrix
   iii. A fixpoint at the bottom of the potential, and
   iv. Nicely spaced curves demonstrating the different types of motion (circulating vs. oscillating).

   (b) In class we discussed how to find the period (and thus frequency) of bound motion in a potential, which involves integrating

   $$T = 2\sqrt{\frac{m}{2}}\int_{x_-}^{x^+}\frac{dx'}{\sqrt{E - V(x')}}\,, \tag{29}$$

   where $x_\pm$ are the turning points of the bound motion for a given energy E.

   The rigid pendulum above is a type of *anharmonic* oscillator, since the frequency of oscillation depends on the amplitude or energy. For this system (letting $m = L = g = 1$), compute the period or frequency for any bound oscillation. You are free to use numerical tools such as Mathematica's `NIntegrate`, or the python package `scipy`. Extra feel-good points for making a plot of period (or frequency) as a function of energy (or amplitude of oscillation). Also extra feel-good points if you get an analytical result which can be evaluated with specialized methods, instead of general integration methods. Super extra feel-good points for implementing your own numerical quadrature routine to perform the integration.