UNIVERSITY OF MISSISSIPPI
Department of Physics and Astronomy
Advanced Mechanics I (Phys. 709) — Prof. Leo C. Stein — Fall 2018

Problem Set 1 — SOLUTIONS

Due: Friday, Aug. 31, 2018, by 5PM

Here, FW = Fetter and Walecka, and HF = Hand and Finch. Reading FW §1.3 and §1.4 will be helpful for some of the problems. As with research, feel free to collaborate and get help from each other! But the solutions you hand in must be your own work.

1. FW 1.2

   Solution:

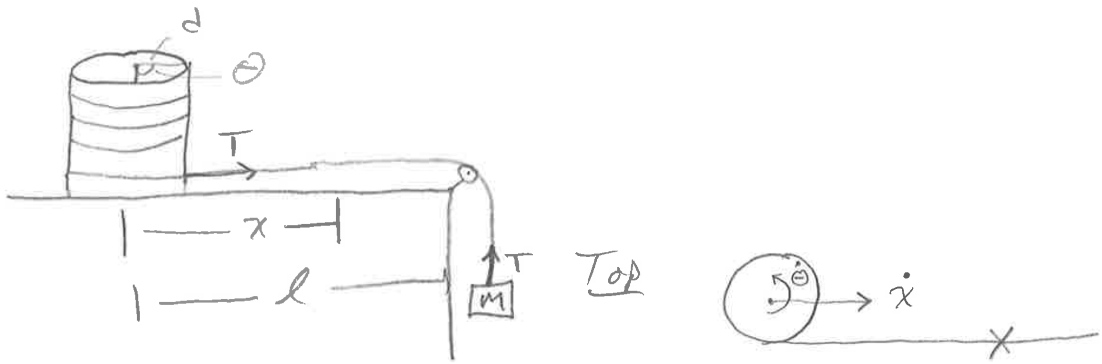

Figure 1: Geometry of FW1.2

Let $x(t)$ be the horizontal displacement (leftward) measured from the center of the cylinder at time $t = 0$, let $\theta(t)$ be the angle by which the cylinder has rotated since that time, and let $z(t)$ be the vertical displacement (downward) of the mass hanging off of the table since time $t = 0$. Then we have the constraint that

$$z(t) = x(t) + \theta(t) * d/2 \,. \tag{1}$$

Consider the angular momentum about a vertical (z) axis, with origin centered over the down-going mass. The down-going mass does not contribute to L_z since its momentum is vertical, and since $\vec{L} = \vec{r} \times \vec{p}$, vertical linear momentum has no contribution to angular momentum about the vertical. The tension on the string is causing the cylinder to rotate, but since we chose the origin directly over the string, there is no torque, so total angular momentum is conserved. The total angular momentum only comes from the rotation of the cylinder and the translation of its center of mass (which is offset from the origin).

$$L_z = I\omega + (\vec{r}_{cyl} \times \vec{p}_{cyl})_z = (\frac{M}{2}(d/2)^2)\dot{\theta} + (d/2)M\dot{x} = 0 \,, \tag{2}$$

using the momentum of inertia of a cylinder about its vertical axis, and equating to the value at $t = 0$, which is conserved. Thus conservation of angular momentum gives us

$$\dot{x} = \frac{d}{4}\dot{\theta} \,. \tag{3}$$

Plugging this into the time derivative of Eq. (1) gives

$$\dot{z} = 3\dot{x} \,. \tag{4}$$

We can immediately integrate this, and since we chose both of these values to vanish at $t = 0$, we have that $z = 3x$. Thus when the cylinder translates by $x_f = \ell$ to the edge of the table, the mass m has fallen by $z_f = 3\ell$.

Now use conservation of energy. The total energy is

$$E = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}m\dot{z}^2 - mgz \tag{5}$$

(the sign of the last term is due to defining z as increasing downward). The initial energy vanishes, so the final energy must as well. Putting everything in terms of z, the final energy is

$$0 = E_f = \frac{1}{2}M(\dot{z}_f/3)^2 + \frac{1}{2}(\frac{M}{2}(d/2)^2)(4\dot{z}_f/3d)^2 + \frac{1}{2}m\dot{z}_f^2 - mg * 3\ell\,. \tag{6}$$

Solving for $\dot{z}$ gives

$$\boxed{\dot{z}_f = \sqrt{\frac{6g\ell}{1 + \frac{M}{3m}}}}\,. \tag{7}$$

If $M \to 0$, we recover free-fall, with $\dot{z}_f = \sqrt{2g(3\ell)}$. If $M \to \infty$, the cylinder becomes fixed, and $\dot{z}_f \to 0$

2. FW 1.4

Solution:

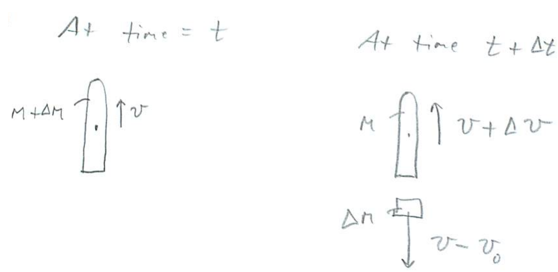

Figure 2: Geometry of FW1.4

The difference in vertical momentum before and after time Δt is from the force of gravity,

$$P_z(t + \Delta t) - P_z(t) = -M(t)g\Delta t\,. \tag{8}$$

Here, the mass as a function of time can be found as $M(t) = m_0(1 - t/\tau)$ since the mass loss rate is constant. Now plug in the total momenta before and after,

$$M(v + \Delta v) + \Delta m(v - v_0) - (M + \Delta m)v = -Mg\Delta t\,. \tag{9}$$

The infinitesimal mass is $\Delta m = m_0/\tau\Delta t$.

If we are cavalier physicists and divide through by Δt and turn this into a differential equation for dv/dt, we find

$$\frac{dv}{dt} = \frac{m_0}{\tau}\frac{v_0}{M(t)} - g \tag{10}$$

$$\frac{dv}{dt} = \frac{v_0}{\tau - t} - g\,. \tag{11}$$

Integrate this with respect to time,

$$\int_0^t \frac{dv}{dt'}dt' = v(t) - v(0) = -gt + \int_0^t \frac{v_0 dt'}{\tau - t'} \tag{12}$$

$$v(t) = -v_0 \ln(1 - t/\tau) - gt\,. \tag{13}$$

Integrate again to get the vertical position $y(t)$,

$$y(t) = -\frac{1}{2}gt^2 - v_0 \int_0^t \ln(1 - t'/\tau)dt' \,. \tag{14}$$

Recall that antiderivative $\int \ln x dx = x \ln x - x$.

Putting it all together, the final result is

$$\boxed{y(t) = \underbrace{v_0 t - \frac{1}{2}gt^2}_{\text{free fall piece}} + \underbrace{v_0(\tau - t)\ln(1 - t/\tau)}_{\text{due to rocket}}} \,. \tag{15}$$

Sanity checks: if $t \ll \tau$, $\ln(1+x) \approx x$, so we end up with $y(t) \approx v_0 t^2/\tau - \frac{1}{2}gt^2$, where height grows quadratically with time. Meanwhile as $t \to \tau$, we need to use that $\lim_{x\to 0} x \ln x \to 0$ (any polynomial goes to zero faster than a log blows up). Here x is proportional to $(\tau - t)$. Then we find that $y(t) \approx v_0\tau - \frac{1}{2}g\tau^2$, as fuel runs out, just get free-fall motion.

3. FW 1.6

Solution:

(a) Working in polar coordinates, Newton's second law for a central force is $m\ddot{\mathbf{r}} = \hat{\mathbf{r}} f(r)$. What is $\ddot{\mathbf{r}}$? Using the product rule,

$$\mathbf{r} = r\hat{\mathbf{r}} \tag{16}$$
$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}} \tag{17}$$
$$\ddot{\mathbf{r}} = \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\hat{\mathbf{r}}} + r\ddot{\hat{\mathbf{r}}} \tag{18}$$

Now resolve $\hat{\mathbf{r}}$ into $\hat{x}, \hat{y}$ components and find its time derivatives,

$$\hat{\mathbf{r}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}} \tag{19}$$
$$\dot{\hat{\mathbf{r}}} = -\sin\phi\dot{\phi}\hat{\mathbf{x}} + \cos\phi\dot{\phi}\hat{\mathbf{y}} = \dot{\phi}\hat{\boldsymbol{\phi}} \tag{20}$$
$$\ddot{\hat{\mathbf{r}}} = \ddot{\phi}\hat{\boldsymbol{\phi}} + \dot{\phi}\dot{\hat{\boldsymbol{\phi}}} \tag{21}$$

And similarly for $\hat{\boldsymbol{\phi}}$ (now we only need one time derivative),

$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} \tag{22}$$
$$\dot{\hat{\boldsymbol{\phi}}} = -\cos\phi\dot{\phi}\hat{\mathbf{x}} - \sin\phi\dot{\phi}\hat{\mathbf{y}} = -\dot{\phi}\hat{\mathbf{r}} \tag{23}$$

Plugging all of this algebra into Eq. (18), we have

$$\boxed{\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\boldsymbol{\phi}}} \,. \tag{24}$$

Now simply equate coefficients of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ in $m\ddot{\mathbf{r}} = \hat{\mathbf{r}} f(r)$ to arrive at the two desired equations.

(b) Multiply the EOM that's the component of $\hat{\boldsymbol{\phi}}$ by r, and then notice that it can be written as a total derivative:

$$0 = (2\dot{r}\dot{\phi} + r\ddot{\phi}) \tag{25}$$
$$0 = r(2\dot{r}\dot{\phi} + r\ddot{\phi}) \tag{26}$$
$$0 = \frac{d}{dt}\left(r^2\dot{\phi}\right) \tag{27}$$

Therefore we find a constant of motion, with the traditional value $r^2\dot{\phi} = \ell/m = \text{const}$. Plugging this into the radial equation gives

$$m\ddot{r} - mr\left(\frac{\ell}{mr^2}\right)^2 = f(r) = -U'(r)\,, \tag{28}$$

where we have taken the force to be conservative, due to the potential $U(r)$. Now move the angular momentum term to the other side, and note that it can be written as an r-derivative of $-\ell^2/2mr^2$. Therefore we can define an effective potential $U_{\text{eff}} = U(r) + \ell^2/2mr^2$. Finally multiply this equation by $\dot{r}$ and note that it can be written as a total derivative,

$$0 = (m\ddot{r} + U'_{\text{eff}}(r)) \tag{29}$$

$$0 = \dot{r}(m\ddot{r} + U'_{\text{eff}}(r)) \tag{30}$$

$$0 = \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r)\right)\,. \tag{31}$$

Once again we find another constant of motion, which this time we call the energy, $E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r)$.

4. FW 1.9

Solution: As with any central potential problem, the effective potential is

$$V_{\text{eff}} = V(r) + \frac{\ell^2}{2mr^2} = -\frac{\lambda}{r}e^{-r/a} + \frac{\ell^2}{2mr^2}\,. \tag{32}$$

This potential looks like it has many parameters: a, λ, ℓ, m. The problem also has an energy E which is another parameter. However, by scaling time, we can set $m = 1$ in the kinetic term (do you see how?). Also, by scaling distances, we can set $a = 1$. Now we are down to the parameters λ, ℓ, E.

First analyze the asymptotic behavior. As $r \to 0$, $e^{-r} \approx 1 - r + \mathcal{O}(r^2)$, so $V(r) \approx -\lambda/r + \mathcal{O}(r^0)$. Meanwhile, the angular momentum barrier goes as $+\ell^2/2r^2$, so the repulsive angular momentum barrier will dominate at small r. Now look at large r. The exponential function decays more quickly than any polynomial, so again the angular momentum piece will dominate.

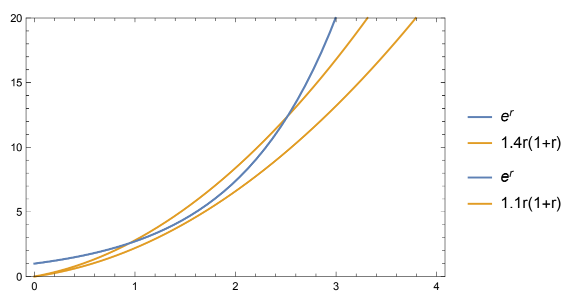

Figure 3: Intersections of e^r and quadratics for finding roots of V'_{eff}.

To understand the intermediate regions, determine any local extrema of V_{eff} by looking at the derivative (with $m = 1 = a$),

$$\frac{dV_{\text{eff}}}{dr} = \frac{-\ell^2}{r^3} + \frac{\lambda e^{-r}(1+r)}{r^2}\,. \tag{33}$$

Extrema have $V' = 0$, or

$$e^r = \frac{\lambda}{\ell^2}r(1+r)\,. \tag{34}$$

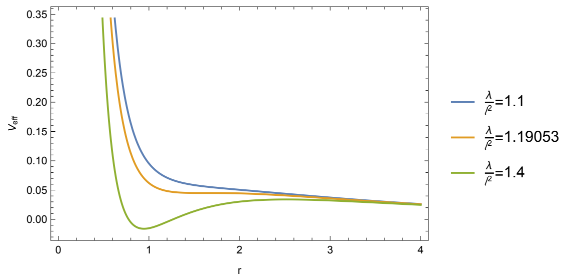

Figure 4: Shape of the effective potential for different values of λ/ℓ^2.

That is, we have a quadratic with zeros at $r = 0$ and $r = -1$, and we want to know if it intersects the exponential e^r for any positive values of r. It can intersect 0, 1, or 2 times: examine plots of these two functions in Fig. 3. The critical case of a single root is when that root is a double root, so the second derivative also vanishes. If we set both $V'_{\rm eff} = 0$ and $V''_{\rm eff} = 0$ and solve, we find a double root for some value of ℓ when $r_0 = (1 + \sqrt{5})/2 = \phi$, the golden ratio. Now plug this back in to $V'_{\rm eff}(\phi) = 0$ to find the critical value of $(\lambda/\ell^2)_{\rm crit} = (\sqrt{5} - 2)e^{\phi} \approx 1.19$. The three types of potentials, with different values of ℓ, are shown in Fig. 4.

For $\lambda/\ell^2 > (\lambda/\ell^2)_{\rm crit}$, the effective potential has two extrema and can support bound orbits. It supports stable and unstable circular orbits, at respectively the concave-up and concave-down extrema. You can have scattering orbits where the energy is greater than the value of the effective potential at the hump, but you can also have scattering orbits outside of the bound region, which have the same energies as bound orbits, but different initial conditions.

For $\lambda/\ell^2 < (\lambda/\ell^2)_{\rm crit}$, there are only scattering orbits.

For the critical case with $\lambda/\ell^2 = (\lambda/\ell^2)_{\rm crit}$, there are scattering orbits except for the marginally unstable circular orbit at the double root in the potential.

5. **Potentials with scaling properties** (Based on HF1.8). Consider a potential $V(\vec{r}_1, \ldots, \vec{r}_M)$ which has the property (for any positive α)

$$V(\alpha\vec{r}_1, \ldots, \alpha\vec{r}_M) = \alpha^k V(\vec{r}_1, \ldots, \vec{r}_M)\,. \tag{35}$$

This is called a homogeneous function, of degree k. Consider an action with this potential and with standard kinetic terms for the M particles of equal mass

$$S = \int \left[\left(\frac{m}{2} \sum_{i=1}^{M} \frac{d\vec{r}_i}{dt} \cdot \frac{d\vec{r}_i}{dt} \right) - V(\vec{r}_1, \ldots, \vec{r}_M) \right] dt\,. \tag{36}$$

(a) What is the transformed action S' if we simultaneously scale distances and times according to $\vec{r}_i' = \alpha\vec{r}_i, t' = \beta t$, for positive α, β? (Hint: pay attention to time derivative and the integration measure)

Solution: We replace all $\vec{r}_i$ and t with their primed variables, being sure to also make the replacement in the integration measure ($dt' = \beta dt$) and in the derivatives ($\frac{d}{dt'} = \frac{1}{\beta}\frac{d}{dt}$). Then using the relationship between the unprimed and primed variables, we find

$$S' = \int \left[\left(\frac{m}{2} \sum_{i=1}^{M} \frac{\alpha^2}{\beta^2} \frac{d\vec{r}_i}{dt} \cdot \frac{d\vec{r}_i}{dt} \right) - V(\alpha\vec{r}_1, \ldots, \alpha\vec{r}_M) \right] \beta dt\,. \tag{37}$$

Now using the homogeneity from Eq. (35) and distributing, this becomes

$$\boxed{S' = \int \left[\left(\frac{m}{2} \sum_{i=1}^{M} \frac{\alpha^2}{\beta} \frac{d\vec{r}_i}{dt} \cdot \frac{d\vec{r}_i}{dt} \right) - \beta\alpha^k V(\vec{r}_1, \ldots, \vec{r}_M) \right] dt}\,. \tag{38}$$

(b) How should β be related to α in order for the transformed action to yield the same equations of motion?

Solution: Multiplying every term in the action by a constant will yield the same equations of motion. Therefore we want to satisfy

$$\frac{\alpha^2}{\beta} = \beta\alpha^k\,, \tag{39}$$

as this will mean that $S' = S \times \text{const}$. Solving for β gives

$$\boxed{\beta = \alpha^{1-k/2}}\,. \tag{40}$$

(c) What is the value of k for (i) a uniform gravitational potential, (ii) a simple harmonic oscillator, and (iii) the Kepler problem?

Solution:

i A uniform gravitational potential takes the form $V = mgz$. If we send $(x, y, z) \to (\alpha x, \alpha y, \alpha z)$, we find that $V(\alpha\vec{r}) = \alpha V(\vec{r})$, and thus $\boxed{k = 1}$.

ii A simple harmonic oscillator has potential $V = \frac{1}{2}\omega^2\vec{r}\cdot\vec{r}$, which is a quadratic polynomial. Therefore it is homogeneous, with degree $\boxed{k = 2}$.

iii The Kepler potential is of the form $V_{ij} = -Gm_im_j/r_{ij}$. If we send all $\vec{r}_i \to \alpha\vec{r}_i$, then the separation between two particles changes as $r_{ij} \to \alpha r_{ij}$. Thus we find the Kepler potential has homogeneity degree $\boxed{k = -1}$.

(d) What does this mean for oscillation frequencies in a SHO, and orbital frequencies in the Kepler problem?

Solution: In part 5b we found that under a spatial scaling $\vec{r} \to \alpha\vec{r}$ and time scaling $t \to \alpha^{1-k/2}t$, the equations of motion would be the same. Thus we can take a *solution* to the original problem and scale distances by a factor of α, and this would give us a new solution where time is multiplied by the factor $\alpha^{1-k/2}$. This means that frequency is multiplied by the reciprocal, $\omega' = \beta^{-1}\omega = \alpha^{-1+k/2}\omega$. So for the SHO, when you scale distances (oscillation amplitude) by a factor of α, frequencies scale by $\alpha^{-1+2/2} = 1$, i.e. $\boxed{\text{frequency is independent of oscillation amplitude in the SHO}}$. Meanwhile, for the Kepler problem, frequencies would scale by the factor $\alpha^{-1+(-1)/2} = \alpha^{-3/2}$. This recovers Kepler's law that $\boxed{\omega \propto a^{-3/2}}$ where a is a characteristic size of the orbit (specifically, the semi-major axis).

6. **Functionals with more derivatives**. Suppose we have a functional that depends on k derivatives of a function,

$$F[f(x)](x_1, x_2) = \int_{x_1}^{x_2} \mathcal{F}\left(f, \frac{df}{dx}, \ldots, \frac{d^kf}{dx^k}\right) dx\,. \tag{41}$$

Suppose we vary $f(x) = f_0(x) + \epsilon\delta f(x)$ with endpoints fixed, $\delta f(x_1) = 0 = \delta f(x_2)$, and similarly for the first $k-1$ derivatives at the endpoints, $\delta f^{(i)}(x_{1,2}) = 0$ for $i = 1 \ldots k-1$. Then what is the generalization of the Euler-Lagrange equation that results from this functional?

Solution: Recall the definition of the variational derivative,

$$\delta F = \frac{d}{d\epsilon}F[f(x) + \epsilon\delta f(x)]\Big|_{\epsilon=0}\,. \tag{42}$$

We take the derivative with respect to ϵ inside the integral, and apply the chain rule,

$$\delta F = \int_{x_1}^{x_2} \sum_{i=0}^{k} \frac{\partial\mathcal{F}}{\partial(f^{(i)})}\frac{d^i\delta f}{dx^i}dx\,. \tag{43}$$

Now we want to move all of the derivatives off of δf by integrating by parts. For example, consider the ith term, and integrate by parts once:

$$\int_{x_1}^{x_2} \frac{\partial \mathcal{F}}{\partial(f^{(i)})} \frac{d^i \delta f}{dx^i} dx = \left[\frac{\partial \mathcal{F}}{\partial(f^{(i)})} \frac{d^{i-1} \delta f}{dx^{i-1}} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial \mathcal{F}}{\partial(f^{(i)})} \right) \frac{d^{i-1} \delta f}{dx^{i-1}} dx \,. \tag{44}$$

First, the boundary term will vanish, because by assumption $\delta f^{(i)}(x_{1,2}) = 0$ for $i = 1 \ldots k-1$. Second, we have converted an ith derivative into an $(i-1)$th derivative acting on δf, multiplied by -1, and put a derivative onto the rest of the integrand.

Now you would integrate by parts again, and the pattern will repeat: vanishing boundary condition, multiplying by -1, and moving one derivative off of δf and onto $\partial \mathcal{F}/\partial(f^{(i)})$. Now you can prove by induction that the result for this one term will be

$$\int_{x_1}^{x_2} \frac{\partial \mathcal{F}}{\partial(f^{(i)})} \frac{d^i \delta f}{dx^i} dx = \int_{x_1}^{x_2} (-1)^i \left(\frac{d^i}{dx^i} \frac{\partial \mathcal{F}}{\partial(f^{(i)})} \right) \delta f dx \,. \tag{45}$$

The only thing left to do is add up all the terms,

$$\delta F = \int_{x_1}^{x_2} \left[\sum_{i=0}^{k} (-1)^i \frac{d^i}{dx^i} \frac{\partial \mathcal{F}}{\partial(f^{(i)})} \right] \delta f dx \,. \tag{46}$$

Since this must vanish for any general variation δf, this gives us the generalized Euler-Lagrange equations,

$$\boxed{0 = \sum_{i=0}^{k} (-1)^i \frac{d^i}{dx^i} \frac{\partial \mathcal{F}}{\partial(f^{(i)})} = \frac{\partial \mathcal{F}}{\partial f} - \frac{d}{dx} \frac{\partial \mathcal{F}}{\partial(f^{(1)})} + \frac{d^2}{dx^2} \frac{\partial \mathcal{F}}{\partial(f^{(2)})} - \ldots} \tag{47}$$

where the sign of each term is determined by the parity of i.