TW exercise 35.13); and the other three degrees of freedom can be used locally n a local inertial frame of the background $g^{(B)}_{\mu\nu}$) to guarantee that

$$h_{0\alpha} = 0, \quad h_{ij} = h^{TT}_{ij} \quad (\text{"local TT gauge"}), \tag{2.35}$$

ere h^{TT}_{ij} is the gravitational-wave field defined, in the background LIF, by

$$R^{(W)}_{i0j0} \equiv - \tfrac{1}{2} h^{TT}_{ij,00} . \tag{2.36}$$

f the background is approximated as flat throughout the wave zone, then one can ntroduce a global inertial frame of $g^{(B)}_{\mu\nu}$ throughout the wave zone and one can im- ose the TT gauge globally. However, if the background is curved, a global TT auge cannot exist (MTW exercise 35.13).

One often knows $h_{\alpha\beta}$ or $\bar{h}_{\alpha\beta}$ in a Lorentz but non-TT gauge and wants to compute ts "gauge-invariant part" h^{TT}_{ij} in some LIF of the background. Such a compu- ation is performed most easily by a "TT projection", which is mathematically quivalent to a gauge transformation (MTW Box 35.1): One identifies the propaga- ion direction n_j in the LIF as the direction in which the wave is varying rapidly on length scale $\lambda\!\!\!^-$). One then obtains h^{TT}_{ij} by discarding all parts of h_{ij} or $\bar{h}_{ij}$ along n_j and by then removing the trace:

$$h^{TT}_{ij} = P_{ia}h_{ab}P_{bj} - \tfrac{1}{2} P_{ij}P_{ab}h_{ab} = (\text{same expression with } h_{ab} \to \bar{h}_{ab}), \tag{2.37}$$

where $P_{ab} = \delta_{ab} + n_a n_b$. WARNING: This projection process gives the correct answer only in an LIF of the background and only if $h_{\mu\nu}$ is in a Lorentz gauge.

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Exercise 10. Show that the infinitesimal coordinate change (2.30) produces the claimed gauge change of $h_{\mu\nu}$. Show further that the Riemann tensor of the waves is correctly given by (2.32) in any gauge, and that this Riemann tensor is invariant under gauge changes (2.30).

Exercise 11. Show that a gauge change with $\xi_{\alpha|\mu}{}^{\mu} = 0$ can be used to make a Lorentz-gauge $h_{\mu\nu}$ trace-free globally (eq. 2.34) and TT locally (eq. 2.35). Show further that the TT projection process (2.37) produces the same result as this gauge transformation.

2.4.3 Absorption and dispersion of waves by matter

When electromagnetic waves propagate through matter (e.g., light through water, radio waves through the interplanetary medium), they are partially absorbed and partially scatter off charges; and the scattered and primary waves superpose in such a way as to change the propagation speed from that of light in vacuum ("Dispersion"). A typical model calculation of this absorption and dispersion in- volves electrons of charge e, mass m, and number density n, each bound to a "lattice point" by a 3-dimensional, isotropic, damped, harmonic-oscillator force:

$$\ddot{\underset{\sim}{z}} + (1/\tau_*)\dot{\underset{\sim}{z}} + \omega_o^2 \underset{\sim}{z} = (e/m)\underset{\sim}{E} = - (e/m)\dot{\underset{\sim}{A}}, \tag{2.38a}$$

where $\underset{\sim}{A}$ is the vector potential in transverse Lorentz gauge and a dot denotes $\partial/\partial t$. These electrons produce a current density $\underset{\sim}{J} = ne(d\underset{\sim}{z}/dt)$, which enters into Maxwell's equations for wave propagation $\Box \underset{\sim}{A} = -4\pi \underset{\sim}{J}$ to give waves of the form $\underset{\sim}{E} = \underset{\sim}{E}_0 \exp(-i\omega t + i\underset{\sim}{k}\cdot\underset{\sim}{x})$ with the dispersion relation (for weak dispersion)

$$\frac{\omega}{k} = (\text{phase speed}) = 1 - \frac{2\pi n e^2/m}{\omega_o^2 - \omega^2 - i\omega/\tau_*} . \tag{2.38b}$$

This dispersion relation shows both absorption (imaginary part of ω/k) and dispersion (real part), and in real situations either or both can be very large.

When gravitational waves propagate through matter they should also suffer adsorption and dispersion. However, in real astrophysical situations the absorption and dispersion will be totally negligible, as the following model calculation shows. (For previous model calculations similar to this one see Szekeres 1971.)

The best absorbers or scatterers of gravitational waves that man has devised are Weber-type resonant-bar gravitational-wave detectors (§§4.1.2 and 4.1.4). On larger scales, a spherical self-gravitating body such as the earth or a neutron star is also a reasonably good absorber and scatterer (good compared to other kinds of objects such as interstellar gas). Consider, then, as idealized "medium" made of many solid spheres (spheres to avoid anisotropy of response to gravity waves), each of which has quadrupole vibration frequency ω_o, damping time (due to internal friction) τ_*, mass m and radius R. For ease of calculation (and because we only need order of magnitude estimates) ignore the self gravity and mutual gravitational interactions of the spheres, and place the spheres at rest in a flat background spacetime with number per unit volume n. Let h^{TT}_{ij} be the gravitational-wave field and require $\lambda\!\!\!^- > n^{-1/3} > R$. The waves' geodesic deviation force drives each sphere into quadrupolar oscillations with quadrupole moment $\mathcal{I}_{jk}$ satisfying the equation of motion (Exercise 22 in §4.1.4 below)

$$\ddot{\mathcal{I}}_{jk} + (1/\tau_*)\dot{\mathcal{I}}_{jk} + \omega_o^2 \mathcal{I}_{jk} = (1/5) m R^2 \ddot{h}^{TT}_{jk} \tag{2.39a}$$

(analog of the electromagnetic equation 2.38a). As a result of these oscillations each sphere reradiates. The wave equation for h^{TT}_{jk} with these reradiating sources (analog of $\Box \underset{\sim}{A} = -4\pi \underset{\sim}{J}$) is

$$\Box h^{TT}_{jk} \equiv \eta^{\alpha\beta} h^{TT}_{jk,\alpha\beta} = -8\pi n \ddot{\mathcal{I}}_{jk} \tag{2.39b}$$

(Exercise 12). By combining equations (2.39a,b) and assuming a wave of the form $h_{jk} \propto \exp(-i\omega t + i\underset{\sim}{k}\cdot\underset{\sim}{x})$ we obtain the gravitational-wave dispersion relation

$$\frac{\omega}{k} = (\text{phase speed}) = 1 - \frac{(4\pi/5) n m R^2 \omega^2}{\omega_o^2 - \omega^2 - i\omega/\tau_*} . \tag{2.39c}$$

To see that the absorption and dispersion are negligible, compare the length scale $\ell = |(1 - \omega/k)\omega|^{-1}$ for substantial absorption or for a phase shift of $\sim \pi/2$ with the radius of curvature of spacetime produced by the scatterers (i.e., the maximum size that the scattering region can have without curling itself up into a closed universe), $\mathcal{R} = (nm)^{-1/2}$:

$$\frac{\ell}{\mathcal{R}} = \underbrace{\left| \frac{\omega_o^2 - \omega^2 - i\omega/\tau_*}{(4\pi/5)\omega^2} \right|}_{\substack{\gtrsim 1 \text{ off resonance} \\ \sim (1/Q) \text{ on resonance}}} \frac{1}{\underbrace{(nR^3)^{1/2}}_{\lesssim 1} \underbrace{(m/R)^{1/2}}_{< 1/\sqrt{2}} \underbrace{(\omega R)}_{< 1}} . \tag{2.40}$$

Here $Q = 1/\omega\tau_*$ is the quality factor of a scatterer, $nR^3 \lesssim 1$ because the scatterers cannot be packed closer together than their own radii, $m/R < 1/2$ because a scatterer cannot be smaller than a black hole of the same mass, and $\omega R = R/\lambda\!\!\!^- < 1$

was required to permit a geodesic-deviation analysis (see above). In the most extreme of idealized universes $\mathcal{L}/\mathcal{R}$ can be no smaller than unity off resonance (dispersion) and $1/Q$ on resonance (absorption); and such extreme values can be achieved only for neutron stars or black holes ($m/R \sim 1$) packed side by side ($nR^3 \sim 1$) with $R \sim \bar{\lambda}$. In the real universe, $\mathcal{L}/\mathcal{R}$ will always be $\ggg 1$; i.e., absorption and dispersion will be negligible regardless of what material the waves encounter and regardless of how far they propagate through it.*

For this reason, henceforth in discussing wave propagation through astrophysical matter (e.g., the interior of the Earth or Sun) I shall approximate $\bar{h}_{\mu\nu|\alpha}{}^{\alpha} = -16\pi\delta T_{\mu\nu}$ by $\bar{h}_{\mu\nu|\alpha}{}^{\alpha} = 0$. The matter will influence wave propagation only through the background curvature it produces (covariant derivative "$|$"), not through any direct scattering or absorption ($\delta T_{\mu\nu}$); see §2.6.1 below.

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Exercise 12. For non-self-gravitating matter in flat spacetime and in Lorentz coordinates, show that $T^{\alpha\beta}{}_{,\beta} = 0$ implies $T^{jk} = (1/2)(\rho x^j x^k)_{,00}$ + (perfect spatial divergence), where ρ is mass density. Average this over a lattice of oscillating spheres with number density $n > \bar{\lambda}^{-3}$ to get $T^{jk} = (1/2)n\ddot{I}_{jk}$, where $I_{jk} = \int \rho x^j x^k d^3x$ is the second moment of the mass distribution of each sphere. Passing gravitational waves excite the oscillations in accord with equation (2.39a) (result to be proved in Exercise 22). These oscillations involve no volume changes, so $\ddot{I}_{jk} = \ddot{\mathcal{I}}_{jk}$ = (trace-free part of $\ddot{I}_{jk}$); moreover, equation (2.39a) shows that $\ddot{\mathcal{I}}_{jk}$ is transverse and traceless. Show that this permits TT gauge to be imposed in the field equations (2.29) in the presence of the oscillating, reradiating spheres (usually it can be imposed only outside all sources), and that the resulting field equations reduce to (2.39b). Then derive the gravitational-wave dispersion relation (2.39c) and the estimate (2.40) of the effects of dispersion and absorption.

2.4.4 Scattering of waves off background curvature, and tails of waves

A self-gravitating body of mass m and size R will typically generate gravitational waves with reduced wavelength

$$\bar{\lambda} \sim (R^3/m)^{\frac{1}{2}} \sim \mathcal{R}_s = \text{(radius of curvature of spacetime near source)}. \tag{2.41}$$

If the body has strong self gravity, $m/R \sim 1$ (neutron star or black hole), then $\bar{\lambda} \sim \mathcal{R}_s$ in the innermost parts of the wave zone; and the curvature coupling terms must be retained in the first-order Einstein equations (2.29). These terms cause the waves to scatter off the background curvature; and repetitively backscattered waves superimposing on each other produce a gravity-wave "tail" that lingers near the source long after the primary waves have departed, dying out as $t^{-(2\ell+2)}$ for waves of multipole order ℓ. See, e.g., Price (1972) for a more detailed discussion, and Cunningham, Price, and Moncrief (1978) for an explicit example.

I regard these backscatterings and tails as part of the wave generation problem and as irrelevant to the problem of wave propagation. In fact, I have defined the inner edge of the "local wave zone" to be so located that throughout it, and throughout the wave propagation problem, $\bar{\lambda} \ll \mathcal{R}$ and backscatter and tails can be ignored (eq. 1.7 and associated discussion).

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*For description of a physically unrealistic but conceivable material in which dispersion is so strong that it actually reflects gravitational waves see Press (1979).

2.4.5 The stress-energy tensor for gravitational waves

Gravitational waves carry energy and momentum and can exchange them with matter, e.g., with a gravitational-wave detector. Isaacson (1968) (see also §35.15) has quantified this by examining nonlinear corrections to the wave-propagation equation (2.31b). In this section I shall sketch the main ideas of his analysis.

Consider a gravitational wave with $\bar{\lambda} \ll \mathcal{L} \lesssim \mathcal{R}$, and expand the metric of the full spacetime in a perturbation series

$$\underset{1,\mathcal{L}}{g_{\mu\nu} = g^{(B)}_{\mu\nu}} + \underset{\mathcal{A},\bar{\lambda}}{h_{\mu\nu}} + \underset{\mathcal{A}^2,\bar{\lambda}}{j_{\mu\nu}} + \dots \;. \tag{2.42a}$$

Below each term I have written the characteristic magnitudes (1, $\mathcal{A}$, $\mathcal{A}^2$) of the metric components, and the lengthscales ($\mathcal{L}$, $\bar{\lambda}$) on which they vary in the most "steady" of coordinate systems. Note that $j_{\mu\nu}$ is a nonlinear correction to the propagating waves. By inserting this perturbation series into the standard expression (MTW eqs. 8.47-8.49) for the Einstein curvature tensor $G_{\mu\nu}$ in terms of $g_{\mu\nu}$ and its derivatives, and by grouping terms according to their magnitudes and their lengthscales of variation, one obtains

$$G_{\mu\nu} = \underset{\le 1/\mathcal{R}^2,\mathcal{L}}{G^{(B)}_{\mu\nu}} + \underset{\mathcal{A}/\bar{\lambda}^2,\bar{\lambda}}{G^{(1)}_{\mu\nu}(h)} + \underset{\mathcal{A}^2/\bar{\lambda}^2,\bar{\lambda}}{G^{(2)}_{\mu\nu}(h)} + \underset{\mathcal{A}^2/\bar{\lambda}^2,\bar{\lambda}}{G^{(1)}_{\mu\nu}(j)} + \dots \;. \tag{2.42b}$$

Here $G^{(B)}_{\mu\nu}$ is the Einstein tensor of the background metric $g^{(B)}_{\mu\nu}$; $G^{(1)}_{\mu\nu}$(h or j) is the linearized correction to $G_{\mu\nu}$ (MTW eq. 35.58a, trace-reversed); and $G^{(2)}_{\mu\nu}(h)$ is the quadratic correction (MTW eq. 35.58b), trace-reversed).

Isaacson splits the Einstein equations into three parts: a part which varies on scales $\mathcal{L}$ (obtained by averaging, "$\langle\ \rangle$", over a few wavelengths)

$$G^{(B)}_{\mu\nu} = 8\pi\left(T^{(B)}_{\mu\nu} + \langle T^{(2)}_{\mu\nu}\rangle + T^{(W)}_{\mu\nu}\right), \quad T^{(W)}_{\mu\nu} \equiv -(1/8\pi)\,\langle G^{(2)}_{\mu\nu}(h)\rangle\,; \tag{2.43a}$$

a part of magnitude $\mathcal{A}/\bar{\lambda}^2$ which varies on scales $\bar{\lambda}$ and averages to zero on larger scales

$$G^{(1)}_{\mu\nu}(h) = 8\pi\, T^{(1)}_{\mu\nu} \Longleftrightarrow \bar{h}_{\mu\nu|\alpha}{}^{\alpha} = -16\pi\, T^{(1)}_{\mu\nu} \quad \text{in Lorentz gauge;} \tag{2.43b}$$

and a part of magnitude $\mathcal{A}^2/\bar{\lambda}^2$ which varies on scales $\bar{\lambda}$ and averages to zero on larger scales

$$G^{(1)}_{\mu\nu}(j) = -\,G^{(2)}_{\mu\nu}(h) + \langle G^{(2)}_{\mu\nu}(h)\rangle + 8\pi\left(T^{(2)}_{\mu\nu} - \langle T^{(2)}_{\mu\nu}\rangle\right). \tag{2.43c}$$

Here $T^{(B)}_{\mu\nu}$ is the stress-energy tensor of the background; $T^{(1)}_{\mu\nu}$ and $T^{(2)}_{\mu\nu}$ are its first- and second-order perturbations; $T^{(W)}_{\mu\nu} \equiv -(1/8\pi)\langle G^{(2)}_{\mu\nu}(h)\rangle$ is a stress-energy tensor associated with the gravitational waves; and the averaging $\langle\ \rangle$ can be performed in the most naive of manners if the coordinates are sufficiently "steady", but must be performed carefully, by Brill-Hartle techniques (MTW exercise 35.14), if they are not. The "smoothed" field equations (2.43a), together with the contracted Bianchi identities $G^{\mu\nu}_{(B)}{}_{|\nu} \equiv 0$, imply a conservation law for energy and momentum in the presence of gravitational waves:

$$\frac{1}{a_o\Sigma_o} \equiv \frac{1}{R} = \frac{H_o q_o^{\,2}(1+Z)}{-q_o + 1 + q_o Z + (q_o-1)(2q_o Z+1)^{1/2}}$$

$$\approx \frac{H_o}{Z}\left[1 + \tfrac{1}{2}(1+q_o)Z + O(Z^2)\right] \qquad \text{for} \qquad Z \ll 1 \qquad (2.64)$$

$$\approx H_o q_o \qquad \text{for } Z \gg 1 \text{ and } Z \gg 1/q_o$$

(MTW eqs. 29.28-29.33). Here H_o is the Hubble expansion rate; q_o is the deceleration parameter of the universe; Z is the cosmological redshift of the source; and I have assumed zero cosmological constant. For formulas with nonzero cosmological constant see MTW eqs. (29.32).

* * * * *

Exercise 16. Show that for propagation through a Friedmann universe equations (2.55)-(2.58) become (2.59)-(2.63).

2.6 Deviations from geometric optics

I have already discussed in detail several ways that wave propagation can differ from geometric optics: absorption and dispersion by matter (§2.4.3; almost always negligible for gravitational waves), and scattering of waves off background curvature with resulting production of tails (§2.4.4; important primarily near source, but also if waves encounter a sufficiently compact body — e.g., a neutron star or black hole). In this section I shall describe two other nongeometric-optics effects: diffraction and nonlinear interactions of the wave with itself.

2.6.1 Diffraction

As gravitational and electromagnetic waves propagate through the universe, they occasionally encounter regions of enhanced spacetime curvature due to concentrations of matter (galaxies, stars, ...) which produce a breakdown in $ƛ \ll \mathcal{L}$ and/or in $ƛ \ll \mathcal{L}^{(W)}$ and a resulting breakdown in geometric-optics propagation. Such a breakdown is familiar from light propagation, where it is called "diffraction".

Consider, as an example, the propagation of waves through the neighborhood and interior of the sun (Fig. 4), and ignore absorption and dispersion by direct interaction with matter (justified for gravitational waves, §2.4.3; not justified for electromagnetic waves). As they pass near and through the sun, rays from a distant source are deflected and forced to cross each other; i.e., they are

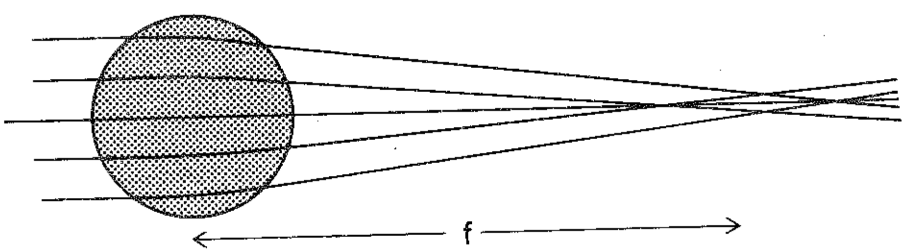

Fig. 4 The rays for geometric-optics wave propagation through the sun.

focussed gravitationally. The dominant source of deflection is the spacetime curvature of the solar core. It produces ray crossing ("caustics") along the optic axis at distances of order (and greater than) the "focal distance"

$$f \sim \frac{\mathcal{L}}{4M/\mathcal{L}} \simeq 20 \text{ AU} . \qquad (2.65)$$

Here $\mathcal{L} \sim 10^5$ km is the inhomogeneity scale of the solar core, $M \sim 0.3\, M_\odot$ is the mass of the solar core, and the value 20 AU comes from detailed calculations with a detailed solar model (Cyranski and Lubkin 1974).

Geometric optics would predict infinite amplification of the waves at the caustics. However, geometric optics breaks down there because it also predicts $\mathcal{L}^{(W)} \to 0$. To understand the actual behavior of the waves near the caustics, think of the waves which get focussed by the solar core as a single wave packet that has transverse dimension $\Delta y \sim \mathcal{L}$ as it passes through the core. The uncertainty principle for waves ($\Delta y \Delta k_y \gtrsim 1$) forces this wave packet to spread in a nongeometric optics manner with a spreading angle

$$\theta_s \sim \Delta k_y / k_x \sim ƛ/\mathcal{L} . \qquad (2.66)$$

This spreading is superimposed on the geometric-optics focussing, and it spreads out the highly focussed waves near the caustics over a lateral scale y_s

$$y_s \sim (ƛ/\mathcal{L}) f \sim (ƛ/4M)\mathcal{L} . \qquad (2.67)$$

If $y_s \ll \mathcal{L}$ (i.e., if $ƛ \ll 4M$) there is substantial focussing: the wave energy density increases near the caustics by a factor $\sim (y_s/\mathcal{L})^2$ and the amplitude increases by $\sim y_s/\mathcal{L} \sim ƛ/4M$. The details of this regime are described by the laws of "Fresnel diffraction". On the other hand, if $y_s \gtrsim \mathcal{L}$ (i.e., if $ƛ \gtrsim 4M$) there is negligible focussing; and the little focussing that does occur is described by the laws of "Fraunhoffer diffraction". For full details see Bontz and Haugan (1981) and references therein.

For the case of the sun the dividing line between substantial focussing and little focussing is $ƛ \sim$ (gravitational radius of sun), i.e., (frequency) $\sim 10^4$ Hz. Since all strong sources of gravitational waves are expected to have $ƛ \gtrsim$ (gravitational radius of source) $\gtrsim$ (gravitational radius of Sun), i.e., (frequency) $\lesssim 10^4$ Hz, they all lie in the "little focussing regime" — a conclusion that bodes ill for any efforts to send gravitational-wave detectors on spacecraft to the orbit of Uranus in search of amplified gravitational waves; cf. Sonnabend (1979).

Far beyond the focal region the geometric optics approximation becomes valid again, except for a smearing of lateral structure of the waves over an angular scale $\sim \theta_s$. For example, ray crossing may produce multiple images of a gravitational-wave source in this region; and those images can be computed by geometric optics methods aside from θ_s-smearing.

2.6.2 Nonlinear effects in wave propagation

Once a gravitational wave has entered and passed through the local wave zone, its nonlinear interactions with itself are of no importance. To see this consider the idealized problem of a radially propagating, monochromatic wave in flat spacetime. At linear order, in spherical coordinates write the wave field as

$$h_{\hat\theta\hat\theta} = -h_{\hat\varphi\hat\varphi} = A_o(\theta,\varphi)\, \frac{ƛ}{r} \cos\left(\frac{t-r}{ƛ}\right) , \qquad (2.68)$$

where hats denote components in an orthonormal, spherical basis. Note that the